STRUCTURE THEOREMS FOR PROJECTIVE MODULES

G.A. CHICAS REYES

Abstract

The present document is a survey of the basic properties of projective modules and some classical structure theorems due to Serre and Bass. In addition, a splitting property for projective modules recently established by Gabber, Liu and Lorenzini is also discussed.

CONTENTS

INTRODUCTION

In this document we study the existence of the following splitting property for projective modules, as established in the paper [GLL15] using methods of algebraic geometry:

THEOREM 0.1: Let A be a ring, and M a projective A-module of constant rank $r > 1$. There exists an A-algebra B that is finite and faithfully flat over A, and such that $M \otimes_A B$ is isomorphic to a direct sum of projective B-modules of rank 1.

The material is organized as follows. Section 1 introduces basic facts about projective modules and some concrete examples. Section 2 presents some classical structure theorems for projective modules established around the 60s and 70s. Afterwards, Section 3 deals with the necessary background of algebraic geometry to present the main result, which is contained in Section 4.

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1 PROJECTIVE MODULES

The purpose of this section is to introduce projective modules and discuss some of its properties. This is stantard material that can be found in many algebra textbooks. Our treatment is mostly based in that of [Bou06], [Lam99], [Lam06] and [IR05].

1.1 *First definitions*

DEFINITION 1.1: An A-module P is *projective* if for every surjective A-linear map $f : M \to N$ and every A-linear map $g : P \to N$ there is a unique A-linear map $h : P \to M$ such that $g = fh$, i.e. the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

There are several other guises for the previous definition. Before studying them, let us prove some useful facts.

LEMMA 1.2: For all A-modules $\{M_\lambda\}_{\lambda\in\Lambda}$ and N there is a natural isomorphism

$$\mathrm{Hom}_A\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\right) \cong \prod_{\lambda\in\Lambda} \mathrm{Hom}_A(M_\lambda, N).$$

Proof. Indeed, additive functors preserve limits. □

LEMMA 1.3: For every collection $\{M_\lambda\}_{\lambda\in\Lambda}$ of A-modules, $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is projective if and only if each M_λ is.

Proof. By Lemma 1.2, the functors $\mathrm{Hom}_A\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, \bullet\right)$ and $\prod_{\lambda\in\Lambda} \mathrm{Hom}_A(M_\lambda, \bullet)$ are isomorphic. Thus $\mathrm{Hom}_A\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, \bullet\right)$ is exact if and only if each $\mathrm{Hom}_A(M_\lambda, \bullet)$ is exact. □

PROPOSITION 1.4: For an A-module P, the following are equivalent:

(a) P is projective.

(b) The functor $\mathrm{Hom}_A(P, \bullet) : A\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is exact.

(c) Every A-linear map onto P has a section.

(d) Every exact sequence $0 \to M \to N \to P \to 0$ in A-**Mod** splits.

(e) P is a direct summand of a free A-module.

Proof.

(a) $\Rightarrow$ (b). The functor $\mathrm{Hom}_A(P, \bullet)$ is already left exact, so the only thing to verify is that for every surjective A-linear map $M \to N$, the induced map $\mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$ is surjective. But this is just a rewording of Definition 1.1.

(b) $\Rightarrow$ (c). Let $M \twoheadrightarrow P$ be an A-linear map. Then by (b) there exists a map $s : P \to M$ such that $fs = \mathrm{id}_P$, which is precisely what we want.

(c) $\Rightarrow$ (d). This is one of the avatars of the well-known *splitting lemma* for exact sequences.[1]

[1] *Cf. [HS97], p.24.*

(d) $\Rightarrow$ (e). For every A-module P we can always find a surjective A-linear map $\phi : N \to P$ with N free.[2] By (d), the exact sequence $0 \to \operatorname{Ker}\phi \to N \xrightarrow{f} P \to 0$ splits, so that P is a direct factor of the free module N.

(e) $\Rightarrow$ (a). First assume that $P = \bigoplus_{\lambda\in\Lambda} A_\lambda$ is free. Let $f : Q \twoheadrightarrow R$ and $g : P \to R$ be A-linear maps and take an A-basis $\{a_\lambda\}_{\lambda\in\Lambda}$ for P. Since f is surjective, we can lift each a_λ to some element $b_\lambda \in Q$. Then we can define an A-linear map $h : P \to Q$ by sending each a_λ to b_λ and extending by A-linearity; clearly $fh = g$, as desired.

In the general case, Let $P \cong M \oplus N$ with M free and let f and g be as before. The map g can be extended to $\tilde{g} : M \to R$, $(p, n) \mapsto g(p)$. As M is free, $\tilde{g}$ can be lifted to $\tilde{h} : M \to Q$. Then $h = \tilde{h}|_P : P \to Q$ is an A-linear map satisfying $fh = g$, whence P is projective. $\square$

[2] *For example, we can take N to be the free A-module generated by the elements of P.*

Now let us turn our attention to some examples of projective modules.

EXAMPLE 1.5:

- Free modules are projective: we already verified this in the part (e) $\Rightarrow$ (a) of the previous proof. This is a fundamental example.
- Conversely, not every projective module is free: Let A_1 and A_2 be two nonzero rings, and regard them as $A_1 \times A_2$-modules via the canonical projections. Then A_1 and A_2 are projective over $A_1 \times A_2$ because $A_1 \oplus A_2 \cong A_1 \times A_2$ (Lemma 1.3); however, they are not free over $A_1 \times A_2$ since, for example, $(0, 1)A_1 = 0$.[3]
- Over a field k, linear algebra tells us that every k-vector space (i.e. every k-module) has a basis; thus all k-vector spaces are projective over k.
- Let n be a positive integer. Being a torsion abelian group, $\mathbb{Z}/n\mathbb{Z}$ is not free over $\mathbb{Z}$ and hence is not projective. As we already know that free $\mathbb{Z}$-modules are projective, the structure theorem for finitely generated abelian groups and Lemma 1.3 imply that a finitely generated abelian group is projective if and only if it is free. More generally, a similar characterization can be obtained for modules over a principal ideal domain using the analogous structure theorem.[4]
- Let G a finite abelian group and k a field whose characteristic does not divide $\#G$. Then by Maschke's theorem from representation theory we know that the group algebra $k[G]$ is semisimple; it is also commutative since G is abelian. Hence by Wedderburn's theorem $k[G]$ is isomorphic to a finite product of fields, and consequently every $k[G]$-module is projective.
- The $M_n(A)$-module A^n is projective but not free. Over any semisimple ring, every module is projective, but only the trivial ideas (zero and whole ring) are free.

[3] *The attentive reader may notice that $A_1 \times A_2$ has the unpleasant feature of having nonconnected prime spectrum. We will provide a better example in the subsection on invertible and fractional ideals.*

[4] *In fact, the finite generation hypothesis is not necessary: over a principal ideal domain, projective modules coincide with free modules. This is standard fact and its proof will be omitted. Cf. [HS97], p. 26.*

A condition we will frequently impose on modules is that of finite generation. In this case, we have additional characterizations of projective modules.

PROPOSITION 1.6: For a projective A-module P, the following are equivalent:

(a) P is finitely generated.

(b) P is a direct summand of a free module of finite rank.

(c) There exists an exact sequence

$$A^n \to A^n \to P \to 0$$

for some n. In particular P is finitely presented.[5]

[5] *Recall that an A-module M is* finitely presented *if there is an exact sequence $A^m \to A^n \to M \to 0$ for some m, n.*

Proof.

(a) $\Rightarrow$ (b). Let P be generated by n elements; then there is a surjective A-linear map $\pi : A^n \to P$. By Proposition 1.4 (c), this map has a section, so that $P \oplus Q \cong A^n$ with $Q \cong \operatorname{Ker} \pi$.

(b) $\Rightarrow$ (c). Let $P \oplus Q \cong A^n$ for some n and some A-module Q. Let $\psi : A^n \to A^n$ be the composition of the canonical projection $A^n \to Q$ and the canonical inclusion $Q \to A^n$. Then $\operatorname{Im} \psi = \{(0, q) \in P \oplus Q : q \in Q\}$, which is also the kernel of the canonical projection $A^n \to P$. Thus $A^n \to A^n \to P \to 0$ is an exact sequence.

(c) $\Rightarrow$ (a). Immediate. □

PROPOSITION 1.7: The tensor product of two projective modules is projective.

Proof. Let P and Q be two A-modules. We need to prove that the functor $\operatorname{Hom}_A(P \otimes_A Q, \bullet)$ is exact. But by the adjunction between the tensor and Hom functors we have an isomorphism of functors

$$\operatorname{Hom}_A(P \otimes_A Q, \bullet) \cong \operatorname{Hom}_A(P, \operatorname{Hom}_A(Q, \bullet)).$$

It follows that $\operatorname{Hom}_A(P \otimes_A Q, \bullet)$ is exact, being the composition of the exact functors $\operatorname{Hom}_A(P, \bullet)$ and $\operatorname{Hom}_A(Q, \bullet)$. □

PROPOSITION 1.8: Projective modules are flat.[6]

Proof. First, notice that free modules are flat since tensor products commute with direct sums. For the same reason, $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is flat if and only if each M_λ is. Consequently a direct summand of a flat module is flat, so the claim follows from Proposition 1.4. □

[6] *Recall that an A-module M is* flat *if the functor $\bullet \otimes_A M$ is exact in A-**Mod**.*

The next paragraph concerns duality for projective modules.

DEFINITION 1.9: The *dual* of a left A-module is the A-module $M^* = \operatorname{Hom}_A(M, A)$. Here A acts on M^* via $a \cdot f : x \mapsto f(x)a$.[7]

[7] *In general, if A is not necessarily commutative, the dual of a left A-module is a* right *A-module.*

We have an evaluation map

$$\begin{aligned} M \times M^* &\longrightarrow A \\ (x, f) &\longmapsto f(x). \end{aligned}$$

This induces a canonical map from M into its *bidual* M^{**}:

$$\begin{aligned} M &\longrightarrow M^{**} \\ x &\longmapsto (\hat{x} : f \mapsto f(x)) \end{aligned}$$

It is a well-known fact from linear algebra that this map is an isomorphism for finite dimensional vector spaces; the same argument applies to free modules of finite rank over any ring. We may wonder whether this property can be extended to not necessarily free modules; it turns outs that projective modules provide a suitable generalization.

PROPOSITION 1.10: For every finitely generated projective A-module P, the dual P^* is a finitely generated projective A-module.

Proof. Indeed, letting $P \oplus Q \cong A^n$ for some Q and n, we have the isomorphisms

$$\begin{aligned} \operatorname{Hom}_A(P, A) \oplus \operatorname{Hom}_A(Q, A) &\cong \operatorname{Hom}_A(P \oplus Q, A) \\ &\cong \operatorname{Hom}_A(A^n, A) \\ &\cong \operatorname{Hom}_A(A, A)^n \\ &\cong A^n. \end{aligned}$$

As $\operatorname{Hom}_A(P, A)$ and $\operatorname{Hom}_A(Q, A)$ are direct factors of a free module of finite rank, it follows that they are finitely generated projective, as desired.[8] □

[8] *Observe that this argument fails if we replace A^n with a free module of infinite rank.*

REMARK 1.11: This fails without finite generation. The reader can consult [Lam07], Exercise 2.6 (p. 26) for a folk example: the $\mathbb{Z}$-module $\prod_{n\in\mathbb{N}} \mathbb{Z} = (\bigoplus_{n\in\mathbb{N}} \mathbb{Z})^*$ is is not projective over $\mathbb{Z}$.

Next we present a fundamental characterization of projective modules, namely the existence of *dual bases*.[9]

[9] *We are committing a mild abuse of notation, since the elements of the "basis" will generate P but in general might not be linearly independent.*

PROPOSITION 1.12: An A-module P is projective if and only if there exist $\{x_i\}_{i\in I} \subseteq P$ and $\{f_i\}_{i\in I} \subseteq P^*$ such that for all $x \in P$:

(a) $f_i(x) = 0$ for all but finitely many i.

(b) $x = \sum_i f_i(x)x_i$.

Proof.
($\Rightarrow$) Let P be projective. Fix a surjective A-linear map $g : F \twoheadrightarrow P$ for some free module $F = \bigoplus_{i\in I} Ae_i$. As P is projective, this map has a section $h : P \to F$. Since the e_i form a basis for F, we can write $h(x) = \sum_{i\in I} f_i(x)e_i$ for some $f_i(x) \in A$. The maps $f_i : P \to A$ thus defined are A-linear, i.e. $f_i \in P^*$ for all i; in addition, by the very definition of basis we must have $f_i(x) = 0$ for all but finitely many i. Finally, applying g yields

$$x = gh(x) = \sum_{i\in I} f_i(x)x_i$$

with $x_i = g(e_i) \in P$. The $\{x_i\}$ and $\{f_i\}$ are the desired elements.
($\Leftarrow$) Let $\{x_i\}$ and $\{f_i\}$ be as described in the statement. Let $F = \bigoplus_{i\in I} Ae_i$ be a free module of basis $\{e_i\}_{i\in I}$ and define a map $g : F \to P, e_i \mapsto x_i$. This is an A-linear map and since the x_i generate P we see that it is also surjective. On the other hand, let $h : P \to F, x \mapsto \sum_{i\in I} f_i(x)e_i$. This is an A-linear map that by condition (b) satisfies $gh = \mathrm{id}_P$. Therefore P is isomorphic to a direct summand of the free module F, and thus is projective. □

Let us specialize the previous result to the case of finitely generated modules.

COROLLARY 1.13: An A-module P is finitely generated projective if and only if there exist $x_1, \dots, x_n \in P$ and $f_1, \dots, f_n \in P^*$ for some n, such that for all $x \in P$,

$$x = \sum_{i=1}^{n} f_i(x)x_i.$$

Proof. Proceed exactly as in Proposition 1.12, but replacing F with a finitely generated free module $\bigoplus_{i=1}^n Ae_i$. □

The existence of dual bases allows us to compare a projective module with its bidual.

PROPOSITION 1.14: For every projective module P, the canonical map $P \to P^{**}$ is injective.

Proof. Choose a dual basis $\{x_i\}$, $\{f_i\}$ for P. Suppose that $x \mapsto (f \mapsto f(x)) = 0 \in P^{**}$. Then $f(x) = 0$ for all $f \in P^{**}$, and in particular $f_i(x) = 0$ for all i. But then part (b) of the previous proposition yields $x = \sum_i f_i(x)x_i = 0$. □

Specializing to the case of finitely generated modules, we recover the analog of duality for finite dimensional vector spaces.

COROLLARY 1.15: For every finitely generated projective module P, the canonical map $P \to P^{**}$ is an isomorphism.

Proof. Let $x_1, \dots, x_n \in P$ and $f_1, \dots, f_n \in P^*$ be a pair of dual bases for P. We claim that $f_1, \dots, f_n$ and $\hat{x}_1, \dots, \hat{x}_n$ form a pair of dual bases for P^*. Indeed, by the definition of a dual basis for P, we have $x = \sum_{i=1}^n f_i(x)x_i$ for all $x \in P$. Applying an arbitrary $f \in P^*$ to both sides of this equality yields $f(x) = \sum_{i=1}^n f_i(x)f(x_i) = \sum_{i=1}^n \hat{x}_i(f)f_i(x)$. Hence $f = \sum_{i=1}^n \hat{x}_i(f)f_i$ and our claim is settled.

From the previous argument we see that the f_i generate P^*, and by the same reasoning we infer that the $\hat{x}_i$ generate P^{**}. It follows that $P \to P^{**}$ is a surjective A-linear map, since each $\hat{x}_i$ has preimage x_i by definition. As we already showed injectivity in Proposition 1.14, we conclude that this map is an isomorphism. $\square$

REMARK 1.16: In general the canonical map $P \to P^{**}$ can fail to be surjective if P is not finitely generated. A typical example, analogous to Remark 1.11, is the $\mathbb{Q}$-vector space $P = \bigoplus_{n\in\mathbb{N}} \mathbb{Q}$. Its dual $P^* = \prod_{n\in\mathbb{N}} \mathbb{Q}$ has uncountable dimension over $\mathbb{Q}$, so the same is true for P^{**}. Consequently, the canonical map $P \to P^{**}$ cannot be an isomorphism.

1.2 *Local characterization*

We start by proving an basic fact.

PROPOSITION 1.17: A finitely generated module P over a local ring A is projective if and only if it is free. More precisely, if $\mathfrak{m}$ is the unique maximal ideal of A, then $\operatorname{rk} P = \dim_{A/\mathfrak{m}} P/\mathfrak{m}P$.

Proof. As free modules are always projective, one direction is trivial. So let P be finitely generated projective. Then $P \otimes_A A/\mathfrak{m} \cong P/\mathfrak{m}P$ is an $A/\mathfrak{m}$-vector space of finite dimension, and there is some basis of the form $x_1 \otimes 1, \dots, x_n \otimes 1$ with $x_1, \dots, x_n \in P$. Let $f : \oplus_{i=1}^n Ae_i \to P$ be an A-linear map sending e_i to x_i. Tensoring with $A/\mathfrak{m}$ gives rise to an $A/\mathfrak{m}$-linear map $(A/\mathfrak{m})^n \to P \otimes_A A/\mathfrak{m}$.[10] Comparing the dimensions of both sides, we see this is an isomorphism, so that $M = \operatorname{Coker} f$ satisfies $M/\mathfrak{m}M \cong M \otimes_A A/\mathfrak{m} = 0$, i.e. $M = \mathfrak{m}M$. As M is also finitely generated over A, we are in a position to apply Nakayama's lemma and deduce that $M = 0$, so that f is surjective.

Next, as P is projective, f must split, i.e. $P \oplus \operatorname{Ker} f \cong A^n$. This implies that $\operatorname{Ker} f$ is finitely generated over A, and in addition $\operatorname{Ker} f = \mathfrak{m}(\operatorname{Ker} f)$. Thus another application of Nakayama's lemma yields $\operatorname{Ker} f = 0$, so f is injective. In conclusion, f is an isomorphism, that is, $P \cong A^n$ is free. $\square$

[10] *Notice that the field $A/\mathfrak{m}$ is flat over A, i.e. the functor $\bullet \otimes_A A/\mathfrak{m}$ preserves exact sequences in A-**Mod**.*

REMARK 1.18: As it was famously established in [Kap58], the finite generation assumption is not necessary: every projective module over a local ring is free. Nevertheless, we will not make use of this result.

Next we need to understand how projective modules behave under localization.

PROPOSITION 1.19: Let A be a ring, S a multiplicative subset of A and P a projective A-module. Then $S^{-1}P$ is a projective $S^{-1}A$-module.

Proof. Recall that the functor $S^{-1} : A\text{-}\mathbf{Mod} \to S^{-1}A\text{-}\mathbf{Mod}, M \mapsto S^{-1}M$ is left adjoint to the functor f^* of restriction of scalars by the canonical map $A \to S^{-1}A$, whence there is a natural isomorphism of functors

$$\operatorname{Hom}_{S^{-1}A}(S^{-1}P, \bullet) \cong \operatorname{Hom}_A(P, f^*(\bullet)).$$

If P is projective, then the right hand side is an exact functor, so that the left hand side is also exact, which means that $S^{-1}P$ is projective over $S^{-1}A$. $\square$

LEMMA 1.20: Let A be a ring, S a multiplicative subset and M, N two A-modules with M finitely presented. Then there is a canonical isomorphism

$$S^{-1}\operatorname{Hom}_A(M, N) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Proof. This is a well-known property of finitely presented modules. See [Eis95], Proposition 2.10 (p. 68). $\square$

The next theorem provides a local characterization of finitely generated projective modules.

THEOREM 1.21: Let A be a ring and P be an A-module. The following are equivalent:

(a) P is a finitely generated projective A-module.

(b) P is finitely presented, and $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of A.

(c) For every $\mathfrak{m} \in \operatorname{Max} A$ there exists $f \in A \backslash \mathfrak{m}$ such that P_f is free of finite rank over A_f.

(d) There exist elements $\{f_i\}_{i \in I} \subseteq A$ generating the unit ideal and such that P_{f_i} is a free A_{f_i}-module of finite rank for all $i \in I$.

Proof.

(a) $\Rightarrow$ (b). Let P be finitely generated projective over A. Then P is finitely presented by Proposition 1.6 (c). On the other hand, we can write $P \oplus Q \cong A^n$ for some Q and n, so that localizing at each maximal ideal $\mathfrak{m}$ of A yields $P_{\mathfrak{m}} \oplus Q_{\mathfrak{m}} \cong A^n_{\mathfrak{m}}$. It follows that $P_{\mathfrak{m}}$ is finitely generated projective over $A_{\mathfrak{m}}$.

(b) $\Rightarrow$ (c). Let $\mathfrak{m}$ be a maximal ideal of A. As $P_{\mathfrak{m}}$ is free of finite rank, say n, we can find A-linear maps $\phi : A^n_m \to P_{\mathfrak{m}}$ and $\psi : P_{\mathfrak{m}} \to A^n_{\mathfrak{m}}$ that are each other's inverse. Exploiting the fact that P is finitely presented over A, we have a canonical isomorphism of $A_{\mathfrak{m}}$-modules $\operatorname{Hom}_{A_{\mathfrak{m}}}(A^n_{\mathfrak{m}}, P_{\mathfrak{m}}) \cong \operatorname{Hom}_A(A^n, P)_{\mathfrak{m}}$. This allows us to write $\phi = \phi'/s$ and $\psi = \psi'/t$ for some A-linear maps $\phi' : A^n \to P$ and $\psi' : P \to A^n$ that are each other's inverse. We see at once that there exist $u, v \in A \backslash \mathfrak{m}$ such that $u\phi'\psi' = ust\,\mathrm{id}_P$ and $v\psi'\phi' = vst\,\mathrm{id}_{A^n}$.[11] Setting $f = stuv \in A \backslash \mathfrak{m}$ we can define A_f-linear maps $\phi'' = tuv\phi'/f : A^n_f \to P_f$ and $\psi'' = suv\psi'/f : P_f \to A^n_f$ that are each other's inverse by construction. Therefore P_f is free of finite rank over A_f.

(c) $\Rightarrow$ (d). As $\mathfrak{m}$ varies, we obtain a collection of elements that is not contained in any maximal ideal, and thus generates the unit ideal as desired.

(d) $\Rightarrow$ (a). We need use the principle of *faithfully flat descent*:[12] if $A \to B$ is a faithfully flat extension, then a finitely generated A-module M is projective if and only if $M \otimes_A B$ is (finitely generated) projective over B. Set $B = \prod_{i \in I} A_{f_i}$ and $M = \prod_{i \in I} P_{f_i} = P \otimes_A B$. For each $i \in I$ we have $F_i \cong P_i \oplus Q_i$ for some A_{f_i}-modules Q_i and F_i with the latter free of finite rank. We lose no generality in assuming that all F_i have the same rank, say r. Then M is a direct factor of the free B-module $F = \prod_{i \in I} F_i \cong B^r$, and thus it is finitely generated projective over B. But it turns out that B is faithfully flat over A, whence $M = P \otimes_A B$ being projective over B is equivalent to P being projective over A.

Thus the only thing left to settle is the faithful flatness of B over A; for this, it suffices to prove that for every $\mathfrak{p} \in \operatorname{Spec} A$ we have $\mathfrak{p}B \neq B$.[13] First, as A_{f_i} is a localization of A, it is flat over A. Being a direct sum of flat modules, B is also flat over A. Now consider a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Then there is i such that $f_i \notin \mathfrak{p}$, for otherwise $\mathfrak{p}$ would contain the unit ideal. This means that $\mathfrak{p}$ corresponds to a prime ideal $\mathfrak{p}A_{f_i}$ in A_{f_i}. Thus $\mathfrak{p}A_{f_i} \neq A_{f_i}$ and in particular $\mathfrak{p}B \subseteq \mathfrak{p}A_{f_i} \times \prod_{j \neq i} A_{f_j} \neq B$.[14] □

[11] *For $p/q \in P_{\mathfrak{m}}$, $p/q = \phi\psi p/q = \phi'\psi'(p)/tsq$. Thus $u(\phi'\psi'(p) - tsp) = 0$ for some $u \in A \backslash m$. The same goes for v.*

[12] *[Bouo6] Ch. II, §5.2.*

[13] *Cf. Ibid., Ch I, §3.1, Proposition 1.*

[14] *Notice that in this argument the finiteness of I is irrelevant.*

COROLLARY 1.22: Every finitely presented and flat module is projective. Consequently a finitely generated module is projective if and only if it is finitely presented and flat.

Proof. Indeed, over a local ring finitely presented flat modules are free. Thus, if P is finitely presented flat over A, then $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ for all and condition (b) of Theorem 1.21 is satisfied. □

REMARK 1.23: If the base ring is Noetherian, finitely presented and finitely generated are equivalent notions, so that for finitely generated modules over a Noetherian ring, flatness and projectivity are equivalent notions. The condition of finite generation cannot be removed: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module that is not free, hence not projective.

Let P be a finitely generated A-module. For every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $P_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module. Since $A_{\mathfrak{p}}$ is a local ring, Proposition 1.17 implies that $P_{\mathfrak{p}}$ is free and we can speak of its rank

$\mathrm{rk}_{A_\mathfrak{p}} P_\mathfrak{p} = n_\mathfrak{p}$ as a free module over $A_\mathfrak{p}$, which was proved in the same proposition to be *finite*.

DEFINITION 1.24: The *rank* of a finitely generated projective A-module at a prime ideal $\mathfrak{p}$ of A is the integer $\mathrm{rk}_\mathfrak{p} P = \mathrm{rk}_{A_\mathfrak{p}} P_\mathfrak{p}$. For an integer $n \geq 0$, we say that P *has (constant) rank* n if $\mathrm{rk}_\mathfrak{p} P = n$ for every prime ideal p of A, and we write $\mathrm{rk}\, P = n$.

It is easy to verify the alternative definition

$$\mathrm{rk}_\mathfrak{p} P = \dim_{\kappa(\mathfrak{p})} P \otimes_A \kappa(\mathfrak{p})$$

where $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is the residue field at $\mathfrak{p}$.

Thus for fixed P we have a function $\mathrm{Spec}\, A \to \mathbb{Z}, \mathfrak{p} \mapsto n_\mathfrak{p}$. We must point out that the only thing we need to define this function is that the rank at each point of $\mathrm{Spec}\, A$ be constant.[15] Nevertheless, projective modules are singled out by the following fundamental property.

[15] *Modules satisfying this property are often called* locally free. *There is another, stronger version of this concept, namely that every point in* Spec *A has an open neighborhood on which the rank of M is finite constant.*

PROPOSITION 1.25: The function $\mathrm{rk} : \mathrm{Spec}\, A \to \mathbb{Z}$ is continuous with respect to the Zariski topology on $\mathrm{Spec}\, A$ and the discrete topology on $\mathbb{Z}$. In particular rk is bounded.

Proof. This amounts to saying that rk is locally constant, which is ensured by Theorem 1.21 (c). As $\mathrm{Spec}\, A$ is quasicompact, it follows that this function is bounded. □

In fact, there is a partial converse for finitely generated modules for which the rank function can be defined:

PROPOSITION 1.26: A finitely generated A-module P is projective if and only if $P_\mathfrak{p}$ is free over $A_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}\, A$, and $\mathrm{rk} : \mathrm{Spec}\, A \to \mathbb{Z}$ is continuous.

Proof. □

COROLLARY 1.27: If $\mathrm{Spec}\, A$ is connected,[16] then every finitely generated projective module over A has constant rank.

[16] *Notice that this is equivalent to A having no nontrivial idempotents, i.e. elements $x \in A$ with $x^2 = x$.*

Proof. This is immediate since rk is locally constant. □

EXAMPLE 1.28: Let A be an integral domain with fraction field K. Then every finitely generated projective A-module P has constant rank equal to $\dim_K P \otimes_A K$.

EXAMPLE 1.29: The rank function need not be continuous for nonprojective modules. For example, consider $A = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z}$.

REMARK 1.30: It is interesting to point out that a projective module of *constant* finite rank is necessarily finitely generated.

In the next proposition we collect some basic facts about the behavior of the rank function.

PROPOSITION 1.31: Let P and Q be finitely generated projective A-modules of constant rank. Then:

(a) $\mathrm{rk}(P \oplus Q) = \mathrm{rk}\, P + \mathrm{rk}\, Q$.

(b) $\mathrm{rk}(P \otimes_A Q) = (\mathrm{rk}\, P)(\mathrm{rk}\, Q)$.

(c) $\mathrm{rk}\, \mathrm{Hom}_A(P, Q) = (\mathrm{rk}\, P)(\mathrm{rk}\, Q)$.

(d) $\mathrm{rk}\, P^* = \mathrm{rk}\, P$.

Proof.

(a) Clear since $(P \oplus Q)_\mathfrak{p} \cong P_\mathfrak{p} \oplus Q_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Max}\, A$.

(b) Let $\mathrm{rk}\, P = m$, $\mathrm{rk}\, Q = n$ and fix $\mathfrak{p} \in \mathrm{Spec}\, A$. Then $(P \otimes_A Q)_\mathfrak{p} \cong P_\mathfrak{p} \otimes_A Q \cong A_\mathfrak{p}^m \otimes_A Q \cong Q_\mathfrak{p}^m$. Thus $\mathrm{rk}_\mathfrak{p}(P \otimes_A Q)_\mathfrak{p} = m\, \mathrm{rk}_\mathfrak{p} Q = mn$. Hence $P \otimes_A Q$ has constant rank mn, as desired.

(c) Keep the notation from (b). As finitely generated projective modules are finitely presented, for $\mathfrak{p} \in \operatorname{Spec} A$ we have $\operatorname{Hom}_A(P,Q)_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, Q_{\mathfrak{p}}) \operatorname{Hom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}^m, A_{\mathfrak{p}}^n) \cong A_{\mathfrak{p}}^{mn}$ as $A_{\mathfrak{p}}$-modules. It follows that $\operatorname{rk}_{\mathfrak{p}} \operatorname{Hom}_A(P,Q) = mn$, so that $\operatorname{Hom}_A(P,Q)$ has constant rank mn.

(d) Immediate from (c). □

PROPOSITION 1.32: Let P be a projective A-module of constant rank n generated by n elements $x_1, \dots, x_n$. Then P is free with basis $x_1, \dots, x_n$.

Proof. We have a surjective A-linear map $f : \bigoplus_{i=1}^n Ae_i \to P, e_i \mapsto x_i$ for $i = 1, \dots, n$. Localizing at each $\mathfrak{p} \in \operatorname{Spec} A$ yields an $A_{\mathfrak{p}}$-linear isomorphism $f_{\mathfrak{p}} : A_{\mathfrak{p}}^n \to P_{\mathfrak{p}}$ since we know that P has constant rank n. Consequently $f_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in \operatorname{Spec} A$, which implies that f itself is an isomorphism. □

1.3 *Invertible modules*

In this paragraph we consider projective modules arising from ring extensions. We fix two rings $A \subseteq B$ throughout.

DEFINITION 1.33: For two A-submodules P and Q of B, we define

$$PQ = \{pq : p \in P, q \in Q\}$$
$$P^{-1} = \{b \in B : bP \subseteq A\}.$$

Clearly, these are also A-submodules of B.

LEMMA 1.34: For an A-submodule P of B, the following are equivalent:

(a) There exists an A-submodule Q of B with $PQ = A$.

(b) $PP^{-1} = A$.

Proof. The direction (b) $\Rightarrow$ (a) is trivial. So suppose that $PQ = A$ as in (a). Then $Q \subseteq P^{-1}$, so that $A = PQ \subseteq PP^{-1} \subseteq A$. Hence $PP^{-1} = A$. □

DEFINITION 1.35: An A-submodule of B satisfying either condition of the previous lemma is said to be *invertible*.

Next we establish some fundamental properties of invertible A-submodules of B.

THEOREM 1.36: For an invertible A-submodule P of B, the following hold:

(a) P is finitely generated projective over A.

(b) For every A-submodule M of B, the canonical map $P \otimes_A M \to PM$ is an A-linear isomorphism.

(c) $P^* \cong P^{-1}$ as A-modules.

(d) P is free over A if and only there is $b \in B$ with $P = Ab$.[17]

[17] *Notice that such b must be invertible.*

Proof.

(a) Let $Q = P^{-1}$, so that $PQ = A$. This implies that there exist p_i, q_i, $i = 1, \dots, n$ such that $\sum_{i=1}^n p_i q_i = 1$. Next, for each i define the maps $f_i : P \to A, p \mapsto pq_i$, which are clearly bilinear and thus belong to P^*. Then

$$p = \sum_{i=1}^n pp_iq_i = \sum_{i=1}^n p_i f_i(p)$$

whence $\{p_i\}$, $\{f_i\}$ form a pair of dual basis for P. By Corollary 1.13, P is finitely generated projective over A.

(b) As the canonical map $P \otimes_A M \to PM$ is clearly surjective, only injectivity remains. Take generators $p_1, \dots, p_n$ for P as in (a); then every element of $P \otimes_A M$ can be written as $\sum_{i=1}^n p_i \otimes m_i$ for some $m_i \in M$, $i = 1, \dots, n$. So suppose that $z = \sum_{i=1}^n p_i \otimes m_i$ is mapped to zero. Then $\sum_{i=1}^n p_i m_i = 0$. Using the relation $\sum_{i=1}^n p_i q_i = 1$ from (a) we see that

$$z = \sum_{i=1}^n \sum_{j=1}^n p_i p_j q_j \otimes m_i = \sum_{j=1}^n p_j \otimes \sum_{i=1}^n p_i q_j m_i.$$

But $\sum_{i=1}^n p_i q_j m_i = q_j \sum_{i=1}^n p_i m_i = 0$. Therefore $z = 0$.

(c) Consider the A-linear map

$$\begin{aligned} Q = P^{-1} &\xrightarrow{\theta} P^* \\ q &\longmapsto (p \mapsto pq) \end{aligned}$$

Let $q \in Q$ be such that $\theta(q) = 0$; this means that $qp = 0$ for all $p \in P$. But $q \in qA = qPQ = 0$, so that $q = 0$ and thus θ is injective. As for surjectivity, keeping the notation of (a) we have $f_i = \theta(q_i)$ for all $i = 1, \dots, n$, so we are done since the f_i generate P^*.

(d) First suppose that $P = Ab$ for some $b \in B$. Then $A = PQ = bQ$, whence $bq = 1$ for some $q \in Q$. Thus $b \in B^\times$ and P is free over A with basis $\{s\}$. Conversely, assume that P is free over A. Since it is finitely generated by (a), this implies that $P \cong A^n$ for some n, so that $Q \cong P^* \cong A^n$. Using (b) and (c) it follows that

$$A = PQ \cong P \otimes_A Q \cong A^n \otimes_A A^n \cong A^{n^2}.$$

For a nonzero ring A, this is possible only if $n = 1$ and thus $P = Ab$ for some $b \in B$. Also, the conclusion holds trivially for the zero ring, so the proof is finished. □

Summing up, invertible modules are a special kind of projective modules. But in fact even more can be said:

PROPOSITION 1.37: Every invertible A-submodule of B has rank 1 as a projective module.

Proof. Write $PQ = A$ for some A-module Q. Localizing yields $P_\mathfrak{p} Q_\mathfrak{p} = A_\mathfrak{p} \subseteq Q(A)_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. Thus $P_\mathfrak{p}$ is invertible and hence finitely generated projective over the local ring $A_\mathfrak{p}$. It follows that $P_\mathfrak{p}$ is a free A_p module, necessarily of rank 1 by part (d) of 1.36. Therefore $\operatorname{rk} P = 1$. □

1.4 *Fractional ideals*

Now we turn our attention to a particularly important case of the previous paragraph, namely that of *fractional ideals*, which have a prominent role in the study of Dedekind domains. Here we give a slightly more general treatment for arbitrary rings (possibly containing zero divisors).

Recall that the *total ring of fractions* of a ring A is the localization $Q(A) = S^{-1}A$ with respect to the multiplicative subset $S \subseteq A$ formed by all the elements of A that are not zero divisors. If A is an integral domain, i.e. it has no zero divisors, then $Q(A)$ is just the usual field of fractions $\operatorname{Frac}(A)$. An important observation is that the canonical map $A \to Q(A)$ is injective by construction. Thus we are in a position to apply the results from the previous paragraph.

DEFINITION 1.38: A *fractional ideal* of A is an A-submodule of $Q(A)$.

Observe that a fractional ideal contained in A is nothing but an ideal in the usual sense, which are sometimes called “integral ideals” in this setting. Thus fractional ideals are akin to “ideals with denominators allowed”.

Given two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, we can define as before the sets $\mathfrak{ab}$ and $\mathfrak{a}^{-1}$, which are also fractional ideals themselves. In addition, let

$$(\mathfrak{b} : \mathfrak{a}) = \{x \in Q(A) : x\mathfrak{a} \subseteq \mathfrak{b}\}.$$

This is also a fractional ideal; notice the special case $(A : \mathfrak{a}) = \mathfrak{a}^{-1}$. Furthermore, if $\mathfrak{a} \subseteq A$ then $\mathfrak{a}^{-1} \supseteq A$, while if $1 \in \mathfrak{a}$ then $\mathfrak{a}^{-1} \subseteq A$.

LEMMA 1.39: For every fractional ideal $\mathfrak{a} \subseteq Q(A)$ with $\mathfrak{a} \cap S \neq 0$ we have a canonical $Q(A)$-linear isomorphism

$$\mathrm{Hom}_A(\mathfrak{a}, Q(A)) \cong Q(A).$$

Proof. Define the A-linear map

$$\begin{aligned} Q(A) &\xrightarrow{\lambda} \mathrm{Hom}_A(\mathfrak{a}, Q(A)) \\ x &\longmapsto (a \mapsto xa) \end{aligned}$$

Let us see that λ is injective. Indeed, assuming that $x, x' \in Q(A)$ are such that $\lambda(x) = \lambda(x')$, the definition of λ implies that $(x - x')a = 0$ for all $a \in \mathfrak{a}$. Choosing a to lie in $\mathfrak{a} \cap S \neq 0$ it becomes invertible in $Q(A)$, so that $x = x'$.

For surjectivity, fix an element $b \in \mathfrak{a} \cap S$ and let $f : \mathfrak{a} \to Q(A)$ be A-linear. For $a \in \mathfrak{a}$, choose $s \in S$ such that $sa \in A$. Then $sbf(a) = f(sba) = saf(b)$, or $s(bf(a) - af(b)) = 0$. As s is invertible in $Q(A)$ we obtain $f(a) = (f(b)b^{-1})a$, i.e. $f = \lambda(f(b)b^{-1})$. □

Given two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, we can consider the collection $\mathrm{Hom}_A(\mathfrak{a}, \mathfrak{b})$ of A-linear maps $\mathfrak{a} \to Q(A)$ whose image is contained in $\mathfrak{b}$. Using the previous lemma, we have a canonical identification

$$\mathrm{Hom}_A(\mathfrak{a}, \mathfrak{b}) \cong (\mathfrak{b} : \mathfrak{a})$$

whenever $\mathfrak{a} \cap S \neq \emptyset$. The particular case of $\mathfrak{a}^{-1} = (A : \mathfrak{a})$ gives us $\mathfrak{a}^* \cong \mathfrak{a}^{-1}$ if $\mathfrak{a} \cap S \neq \emptyset$.

THEOREM 1.40: For a fractional ideal $\mathfrak{a}$, the following are equivalent:

(a) $\mathfrak{a}$ is an invertible A-module.

(b) $\mathfrak{a}$ is projective over A and $\mathfrak{a} \cap S \neq \emptyset$.

In addition, if either condition is satisfied, then $\mathfrak{a}$ is finitely generated over A, and it is free over A if and only if $\mathfrak{a} = Ax$ for some $x \in Q(A)^\times$.

Proof.

(a) ⇒ (b). Suppose that $\mathfrak{a}$ is invertible. By Theorem 1.36 we already know that $\mathfrak{a}$ is finitely generated projective over A. It remains to show that $\mathfrak{a} \cap S \neq \emptyset$. Indeed, take elements $p_i \in \mathfrak{a}$, $q_i \in \mathfrak{a}^{-1}$, $i = 1, \ldots, n$ with $\sum_{i=1}^n p_i q_i = 1$. Then there is $s \in S$ such that $p_i = a_i/s$, $q_i = b_i/s$ for all $i = 1, \ldots, n$ with $a_i, b_i \in A$. As $a_i = sp_i \in \mathfrak{a}$ for all i we have $r^2 = \sum_{i=1}^n a_i b_i \in \mathfrak{a} \cap S$ is the desired element.[1]

(b) ⇒ (a). Assume that $\mathfrak{a}$ is projective over A. Then we can take a pair of dual bases $\{a_i\}_{i \in I}$, $\{f_i\}_{i \in I}$ for $\mathfrak{a}$. We already know that each f_i is given by multiplication by some $b_i \in \mathfrak{a}^{-1}$. Now fix some $s \in \mathfrak{a} \cap S$. As $f_i(s) = b_i s$ is zero for all but finitely many i and s is invertible in $Q(A)$, we see that $b_i = 0$ for all but finitely many i. Thus we may assume that I is finite. Writing

$$s = \sum_{i \in I} a_i f_i(s) = \sum_{i \in I} a_i b_i s$$

and canceling s yields a relation $\sum_{i \in I} a_i b_i = 1$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{a}^{-1}$. Therefore $\mathfrak{a}\mathfrak{a}^{-1} = A$.

The rest of the statement is immediate from Theorem 1.36. □

EXAMPLE 1.41: Let A be a ring. Then $Q(A)$ is never projective over A unless $Q(A) = A$. Indeed, by the previous theorem $Q(A)$ should be finitely generated over A, say by $a_1/d, \ldots, a_n/s$ with $a_1, \ldots, a_n \in A$ and $s \in S$. But $Q(A) = \sum_{i=1}^n Aa_i/s$ would imply that $Q(A) = sQ(A) \subseteq A$, i.e. $Q(A) = A$.

EXAMPLE 1.42: Let k be a field and $A = k[x_1, \ldots, x_n]$, $n \geq 2$. Then the ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ is not invertible;[18] more precisely we have that $\mathfrak{m}^{-1} = A$ and thus $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m} \subsetneq A$. Indeed, let $f/g \in \mathfrak{m}^{-1}$ with f, g coprime and $g \notin k$.

[18] *Of course, if $n = 1$ the ideal $(x) \subseteq k[x]$ is invertible since $1 = x \cdot 1/x \in \mathfrak{m}\mathfrak{m}^{-1}$.*

[1] The equalities $a_i = sp_i$, $b_i = sq_i$ imply that $a_i b_i = r^2 p_i q_i$ and $\sum_{i=1}^n a_i b_i = r^2 \sum_{i=1}^n p_i q_i = r^2$.

By the definition of $\mathfrak{m}^{-1}$ we have $h_i = (f/g)x_i \in A$ for each $i = 1, \ldots, n$. In particular, g is divisible by x_1 and so is $gh_2 = x_2 f$. This means that f is divisible by x_1, which contradicts the assumption that f and g be coprime. Consequently g is not divisible by x_1, so the equality $x_1 f = gh_1$ implies that h_1 is divisible by x_1, whence $f/g = h_1/x_1 \in A$, which is impossible. Summing up, $\mathfrak{m}$ is not invertible and in particular not projective over A.

1.5 *Projective modules over Dedekind domains*

In this paragraph we will study finitely generated projective modules over a Dedekind domain. These are perhaps the simplest rings for which nontrivial projective modules exist, yet a complete classification of them can be given.[19]

Let us start by recalling the following definition.

DEFINITION 1.43: A *Dedekind domain* is a ring A satisfying any of the following equivalent conditions:

1. A is Noetherian, integrally closed of dimension 1.
2. A is Noetherian and $A_\mathfrak{p}$ is a *discrete valuation ring*[20] for every $\mathfrak{p} \in \operatorname{Spec} A$.
3. Every nonzero ideal of A is invertible.
4. Every nonzero ideal of A is a product of maximal ideals.
5. Every nonzero ideal of A is a product of prime ideals.

The equivalence of this statements is well-known and will not be reproduced here.[21]

We start with an observation that follows easily from our previous exposition, but is nevertheless very important.

PROPOSITION 1.44: In a Dedekind domain, fractional ideals and invertible ideals coincide and are always finitely generated projective of rank 1. Conversely, every finitely generated projective module of rank 1 is isomorphic to a fractional (or invertible) ideal.

The first step towards a classification is the following characterization of finitely generated projective modules.

THEOREM 1.45: Let A be a Dedekind domain and M a finitely generated A-module. The following are equivalent:

(a) M is projective.

(b) M is flat.

(c) M is torsion free.

Proof.

(a) $\Rightarrow$ (b). This was already proved in Proposition 1.8.

(b) $\Rightarrow$ (c).

Fact 1: Notice first that M is torsion free if and only if $M_\mathfrak{m}$ is torsion free for every $\mathfrak{m} \in \operatorname{Max} A$. Fact 2: A finitely generated module over a Noetherian ring is flat if and only if every localization at a maximal ideal is free. Fact 3: Over a DVR, torsion free=free.

(c) $\Rightarrow$ (a). Let $K = \operatorname{Frac} A$. We establish the claim by induction on $n = \operatorname{rk} M = \dim_K M \otimes_A K$. For $n = 1$, M is an A-submodule of $M \otimes_A K \cong K$, so that M is isomorphic to a fractional ideal in K. As every fractional ideal in an integral domain is isomorphic to an integral ideal, Theorem 1.40 implies that fractionals ideals are invertible and thus projective. Consequently M itself is projective, and the claim holds for $n = 1$.

Next, assume the result true for some n. Choose $n - 1$ elements of M spanning an $n - 1$-dimensional K-vector subspace in $M \otimes_A K$. These elements

[19] *Over Dedekind domains, infinitely generated projective implies free, so they are not interesting.*

[20] *Meaning…*

[21] *Cf., Peter May's* Notes on Dedekind rings, *available online.*

generate an A-submodule $N \subseteq M$ of rank $n-1$. As K is flat over A, the exact sequence

$$0 \to N \to M \to M/N \to 0$$

is preserved upon tensoring by K, i.e.

$$0 \to N \otimes_A K \to M \otimes_A K \to M/N \otimes_A K \to 0$$

is an exact sequence of K-vector spaces. Comparing dimensions we see that $\operatorname{rk} M/N = \dim_K M/N \otimes_A K = \dim_K M - \dim_K N = 1$. By the same argument as before, M/N is projective and therefore the first exact sequence splits. □

The final result describing projective modules over Dedekind domains is the following.

THEOREM 1.46: Let A be a Dedekind domain and M a finitely generated projective A-module of rank n. Then there exists nonzero fractional ideal I such that

$$P \cong A^{n-1} \oplus I.$$

Furthermore, the image of I in the *class group* of A is uniquely determined.

A full proof of this statement will be omitted for reasons of space, but the interested reader is invited to consult Peter May's *Notes on Dedekind rings*.

2 STRUCTURE THEOREMS FOR PROJECTIVE MODULES

This section is devoted to the study of some classical results on the structure of projective modules. We closely follow the treatment of [Jou83].

2.1 *Free rank of a module*

As a first step towards a general classification of projective modules, in this paragraph we study an extension of the notion of rank to modules that are not necessarily locally free.

Let us start recalling that, in an arbitrary category, a *split monomorphism* is an arrow $f : M \to N$ having a left inverse, i.e. an arrow $r : N \to M$ such that $rf = \mathrm{id}_M$.[22] Notice that this automatically implies that f is a monomorphism. In addition, in an abelian category such as A-**Mod** this condition implies that N is a direct factor of M; this is one of the avatars of the well-known *splitting lemma* for exact sequences.

[22] *Dually, a* split epimorphism *is an arrow having a right inverse. We will not deal with them in this paragraph.*

DEFINITION 2.1:

- Let A be a local ring. The *free rank* of an A-module M is the (possibly infinite) supremum of the set of integers $r \geq 0$ for which there is a split monomorphism $A^r \to M$, that is, M admits a direct free factor of rank r. Notation: $\mathrm{rl}_A(M)$.[23]
- Let A be an arbitrary ring. The *free rank* of an A-module M is the (possibly infinite) infimum of the set $\{\mathrm{rl}_{A_\mathfrak{m}}(M_\mathfrak{m}) : \mathfrak{m} \in \operatorname{Max} A\}$. If no confusion arises, the free rank of M over A will be simply denoted by $\mathrm{rl}(M)$.

[23] *From the French* rang libre.

At this point we can make some simple observations:

- for a projective module (over a connected Spec) the free rank equals the usual rank.
- We have an alternative characterization $\mathrm{rl}_A(M) = \inf\{\mathrm{rl}_{A_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}$.

One would expect this quantity to behave in a similar way to the usual definitions of rank; in particular, we may ask ourselves whether $\mathrm{rl}_A(M \oplus N) = \mathrm{rl}_A(M) + \mathrm{rl}_A(N)$ holds for arbitrary M and N. However, this is certainly false. For a trivial example, which we already saw, we can take the product A of two rings A_1 and A_2. Then A_1 and A_2 are projective A-modules of zero rank (hence zero free rank), whereas the free rank of A over itself is obviously 1. However we do have the following additive property:

PROPOSITION 2.2: For every ring A and every A-module M we have

$$\mathrm{rl}_A(A \oplus M) = 1 + \mathrm{rl}_A(M).$$

The proof of this property is based on the following *cancellation lemma*.

LEMMA 2.3: Let A be a local ring, and M, N be two A-modules. If $A \oplus M \cong A \oplus N$, then $M \cong N$.

Proof. The given isomorphism translates into an exact sequence

$$0 \to N \to A \oplus M \xrightarrow{f} A \to 0$$

where f is given by $(a, m) \mapsto ba + \phi(m)$ for some $b \in A$ and $\phi : M \to A$. Then $N = \operatorname{Ker} f = \{(a, m) \in A \oplus M : ba + \phi(m) = 0\}$.

Now we claim that we can find $m \in M$ such that $u = b + \phi(m) \in A^\times$. Indeed, if $b \in A^\times$ then $m = 0$ suffices; otherwise b belongs to the maximal ideal $\mathfrak{m}$ of A, and upon localizing we see that there is $m \in M$ such that $\phi(m) \in A^\times$.

Using this u the condition defining N can be rewritten as

$$N = \{(a, x) \in A \oplus M : a + \phi(u^{-1}((x - am)) = 0\}.$$

But $(a, x) \mapsto (a, u^{-1}(x - am))$ is an automorphism of $A \oplus M$, whence its image $N' = \{(a, y) \in A \oplus M : a + \phi(y) = 0\}$ may be identified with N. But the projection $N' \to M, (a, y) \mapsto y$ is obviously injective and surjective, hence an A-linear isomorphism. □

2.2 *Serre's theorem*

The first structure theorem that we will present was first established in Serre's seminal paper [Ser58]. It tells us that, under some Noetherian hypothesis, projective modules with "large" rank always decompose into a free part plus a projective factor of rank at most the dimension of the ring.

In the same paper, Serre claims to have been inspired by the correspondence between vector bundles and projective modules.

THEOREM 2.4: Let A be a ring such that $\operatorname{Max} A$ is a Noetherian topological space of finite dimension d, and let M be an A-module such that:

1. M is a direct factor of a direct sum of finitely presented A-modules.
2. $\mathrm{rl}_A(M) > d$.

Then there exists an A-module N and an isomorphism $M \cong A \oplus N$.

It is worth noting that, in practice often works with Noetherian rings, for which one knows that $\operatorname{Max} A$ is a Noetherian topological space. In addition, condition (1) applies to finitely presented and in particular to finitely generated projective modules, which are our main interest.

LEMMA 2.5: Let A be an arbitrary ring and P a finitely generated A-module that is a direct factor of a direct sum of finitely presented A-modules. In addition, let $u : P \to A$ be an A-linear map.

1. The set of all $\mathfrak{p} \in \operatorname{Spec} A$ satisfying that $u_\mathfrak{p} : P_\mathfrak{p} \to M_\mathfrak{p}$ be a split monomorphism is open in the Zariski topology.

2. If $u_{\mathfrak{m}}$ is a split monomorphism for all $\mathfrak{m} \in \operatorname{Max} A$, then u itself is a split monomorphism.

Proof. First we reduce to the case $M = \bigoplus_{i \in I} M_i$ with M_i finitely presented. As P is finitely generated, so is its image under u, and thus the latter intersects finitely many of the M_i, i.e. there is a finite subset $J \subseteq I$ with $u(P) \subseteq \bigoplus_{i \in J} M_j =: M'$. As M' is a direct factor of M, replacing M' by M in the statement we may assume that M is finitely presented.

Now let us prove the first part. The fact that $u_{\mathfrak{p}}$ be a direct monomorphism is equivalent to the existence of a homomorphism $w : M_{\mathfrak{p}} \to P_{\mathfrak{p}}$ with $wu_{\mathfrak{p}} = \mathrm{id}_{P_{\mathfrak{p}}}$. Now recall that for finitely presented M and arbitrary P hom-sets can be "localized", i.e.

$$\operatorname{Hom}_A(M, P)_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, P_{\mathfrak{p}}).$$

Under this identification we can write $w = v/s$, with $v : M \to P$ and $s \notin \mathfrak{p}$. Then $(vu)_{\mathfrak{p}} : P_{\mathfrak{p}} \to P_{\mathfrak{p}}$ is an isomorphism, and consequently vu is invertible on a neighborhood of $\mathfrak{p}$. This implies that the localization of u at each point of this neighborhood is a split monomorphism, which establishes the claim.

As for the second part, it suffices to show that the map

$$\begin{aligned} \theta : \operatorname{Hom}_A(M, P) &\longrightarrow \operatorname{Hom}_A(P, P) \\ f &\longmapsto fu \end{aligned}$$

is surjective. But upon localizing at each $\mathfrak{m} \in \operatorname{Max} A$ we get a surjective map, whence θ itself must be surjective, as desired. □

REMARK 2.6: This result can be strengthened in two ways: first, the base ring can be assumed *semilocal*, i.e. having finitely many maximal ideals, and furthermore one can replace A in $A \oplus M \cong A \oplus N$ by a finitely generated A-module. Cf. Paper by Evans and Bass' book on K-theory.

The following piece of notation will be useful in the sequel. Given a ring A and an A-module M, we say a sequence $(m_1, \ldots, m_n)$ of elements of M is *unimodular* at $\mathfrak{m} \in \operatorname{Max} A$ if the map

$$\begin{aligned} (m_1, \ldots, m_n)_{\mathfrak{m}} : A_{\mathfrak{m}}^n &\longrightarrow M_{\mathfrak{m}} \\ (b_1, \ldots, b_n) &\longmapsto b_1 m_1 + \cdots + b_n m_n \end{aligned}$$

is a split monomorphism. We will also say that a sequence is unimodular on a set $S \subset \operatorname{Max} A$ if it is direct at each $\mathfrak{m} \in S$.

The previous lemma tells us that, in a module M that is a direct factor of a direct sum of finitely presented A-modules, a sequence $(m_1, \ldots, m_n)$ is unimodular on $\operatorname{Max} A$ if and only if the A-linear map $(m_1, \ldots, m_n) : A^n \to M$ is a split monomorphism. We call such everywhere unimodular sequences simply *unimodular*.

For example, $m : A_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is direct if and only if m is a unimodular element of $M_{\mathfrak{m}}$ (cf. later section). In light of the previous remark, one obtains that unimodularity is a local notion!

REMARK 2.7: If $(m_1, \ldots, m_n) : A^n \to M, (b_1, \ldots, b_n) \mapsto b_1 m_1 + \cdots + b_n m_n$ is a split monomorphism, then M has a free direct factor of basis $m_1, \ldots, m_n$.

EXAMPLE 2.8: An element $x \in P$ is unimodular if $\phi(x) = 1$ for some $\phi \in P^*$. Notice that for x unimodular with $\phi(x) = 1$ one has $\phi(ax) = a\phi(x) = a$, so that the A-linear map $h : A \to P, a \mapsto ax$ admits ϕ as retraction, i.e. $\phi h = \mathrm{id}_A$. It follows that h is a split monomorphism and thus we have a decomposition $P \cong Ax \oplus Q$; this is obviously a sufficient condition. We easily see as well that x is unimodular if and only if its *order ideal* $\mathfrak{o}(x) = \{\phi(x) : \phi \in P^*\}$ equals the unit ideal.

LEMMA 2.9: Let A be a ring, M and A-module and $\mathfrak{m} \in \operatorname{Max} A$. Suppose that $(m_1, \ldots, m_n)$ and $(m'_1, \ldots, m'_n)$ are two sequences in M with $m_i - m'_i \in \mathfrak{m}$ for all $1 \leq i \leq n$. Then $(m_1, \ldots, m_n)$ is direct at $\mathfrak{m}$ if and only if $(m'_1, \ldots, m'_n)$ is direct at $\mathfrak{m}$.

Proof. We can assume A local with maximal ideal $\mathfrak{m}$. Denote by u and u' the A-linear maps $A^n \to M$ defined by both sequences. If u is a direct monomorphism, then it has a retraction $v : M \to A^n$, i.e. $vu = \mathrm{id}_M$. The given hypothesis implies that $vu' - \mathrm{id} \in \mathfrak{m}A^n$. From Nakayama's lemma it follows that $vu' : A^n \to A^n$ is surjective, hence bijective, whence u' is a direct monomorphism. □

Here is the final step we need towards the proof of Serre's theorem.

LEMMA 2.10: Let A be a ring whose maximal spectrum is a Noetherian topological space, and M an A-module that is a direct factor of a direct sum of finitely presented A-modules. Let $r = \mathrm{rl}(M)$. Suppose that we are given:

- A closed subset $Y \subseteq X$.
- A sequence $(m_1, \ldots, m_s)$ of elements of M that is direct outside Y.[24]
- A finite subset $\Phi \subseteq Y$, and for each $y \in \Phi$ an element $v_y \in M_y/yM_y \cong M/yM$.

[24] *Notice the constraint $r \geq s$.*

Then for every integer $0 \leq k \leq r - s$ there exist a closed subset $F \subseteq X$ and an element $m_{s+1} \in M$ satisfying the following conditions:

1. $\mathrm{codim}(F, X) \geq k$.
2. The sequence $(m_1, \ldots, m_{s+1})$ is unimodular outside $Y \cup F$.
3. For every $y \in \Phi$, we have $\overline{m_{s+1}} = v_y$ in $M_y/yM_y \cong M/yM$.

Notice how this lemma implies Serre's theorem: we just take $k = r > d$, $s = 0$ and $Y = \emptyset$. As F will satisfy the condition $\mathrm{codim}(F, X) \geq r > k$, necessarily $F = \emptyset$ and thus m_1 is unimodular, whence $m_1 : A \to M$ is a split monomorphism.

Proof. The claim will be shown by strong induction on k, keeping the other variables constant. For $k = 0$ one may take $F = X$. Then it is enough to show the existence of $m \in M$ with $\overline{m} = v_y$ in M_y/yM_y for every $y \in \Phi$. As the $y \in \Phi$ are pairwise coprime[25], the Chinese remainder theorem gives an isomorphism

$$M/(\bigcap_{y\in\Phi} y)M \cong \prod_{y\in\Phi} M/yM,$$

[25] *The original text says* fortement étrangers. *What does this mean?*

from which the existence of m is clear.

Now assume the truth of the statement for $k - 1$ for some $k \geq 1$. As $k - 1 \leq r - s$, the induction hypothesis yields the existence of an element $m'_{s+1} \in M$ and a closed subset $S \subseteq X$ satisfying

- $\mathrm{codim}(S, X) \geq k - 1$.
- $(m_1, \ldots, m_s, m'_{s+1})$ is unimodular outside $Y \cup S$.
- $\overline{m'_{s+1}} = v_y$ in M/yM for every $y \in \Phi$.

We almost have what we want, with the inconvenient that the set S is too big. So let us do the following. For each irreducible component S_i of S not contained in Y, choose an element $y_i \in S_i \backslash Y$ not belonging to any other such component $S_{i'}$. As $\mathrm{rl}(M_{y_i}) \geq r$, for each i there exists $\xi_i \in M$ such that $(m_1, \ldots, m_s, m_{s+1} + \xi_i)$ is unimodular in M_{y_i}. Then applying again the induction hypothesis with respect to $k - 1 \leq r - (s + 1)$, the closed set $Y \cup S \subseteq X$ and the finite set $\Phi \cup \{y_i\}_i$ to obtain a new element $m''_{s+1} \in M$ and a new closed subset $T \subseteq X$ satisfying

- $\mathrm{codim}(T, X) \geq k - 1$.
- $(m_1, \ldots, m_s, m'_{s+1}, m''_{s+1})$ is unimodular outside $Y \cup S \cup T$.
- $\overline{m''_{s+1}} = 0 \in M/yM$ for each $y \in \Phi$, and $\overline{m''_{s+1}} = \xi_i$ in M/y_iM for each i.

The second condition clearly implies that $(m_1, \dots, m_s, m'_{s+1} + am''_{s+1})$ is unimodular outside $Y \cup S \cup T$ for all $a \in A$. Now for each irreducible component T_j of T not contained in $Y \cup S$, choose an element $z_j \in T \backslash (T \cap (Y \cup S))$.

By the Chinese remainder theorem, it is possible to find $a \in A$ such that the following holds:

$$a \equiv \begin{cases} 0 \pmod{y} & \text{for all } y \in \Phi \\ 1 \pmod{y_i} & \text{for all } i \\ 0 \pmod{z_j} & \text{for all } j. \end{cases}$$

Finally, let us set

$$m_{s+1} = m'_{s+1} + am''_{s+1}.$$

By the choice of m'_{s+1} and m''_{s+1} it is clear that $m_{s+1} \equiv v_y \pmod{y}$ for all $y \in \Phi$. Furthermore, $m_{s+1} \equiv m'_{s+1} \pmod{z_j}$, whence the sequence $(m_1, \dots, m_{s+1})$ is congruent to $(m_1, \dots, m_s, m_{s+1})$ modulo z_j. As the latter is unimodular outside $Y \cup S$, hence at z_j, it follows that $(m_1, \dots, m_{s+1})$ is unimodular at z_j. Therefore the set of points of X at which $(m_1, \dots, m_{s+1})$ is not unimodular turns out to be Zariski closed. Denoting by F the union of its irreducible components not contained in Y, we readily see that $(m_1, \dots, m_{s+1})$ is unimodular outside $Y \cup F$, and in addition $F \subseteq S \cup T$ by the very definition of S and T. Every irreducible component of F is contained in an irreducible component of either S or T, but equality cannot hold since the y_i and z_j do not belong to F. In conclusion, $\operatorname{codim}(F, X) \geq k$, as desired. □

2.3 *Bass' cancelation theorem*

The second result that will be presented is due to the American mathematician Hymann Bass, who discovered it in connection to his work on K-theory.

THEOREM 2.11: Let A be a ring such that $\operatorname{Max} A$ is a Noetherian topological space of finite dimension d. Suppose that Q is a finitely generated projective A-module, and M, N are two A-modules satisfying $Q \oplus M \cong Q \oplus N$. If M has a projective factor of free rank greater than d, then $M \cong N$.

Proof. Since Q is finitely generated projective, we have $Q \oplus R \cong A^m$ for some A-module R and some k. Then the hypothesis $Q \oplus M \cong Q \oplus N$ implies that $A^m \oplus M \cong A^m \oplus N$. To prove this it suffices to show that $A \oplus M \cong A \oplus N$ implies $M \cong N$, i.e. we may assume that $Q = A$.

As we saw before, an isomorphism $A \oplus M \cong A \oplus N$ gives rise to a split monomorphism $h : A \to A \oplus M, 1 \mapsto (a, m)$ whose cokernel is isomorphic to N. Once this reduction has been made, the proof is based on the following lemma.

LEMMA 2.12: Under the hypothesis of the previous theorem, there exists $m' \in M$ such that $m + am'$ is unimodular.

Once the lemma is proved, we will be able to proceed as follows. From the unimodularity of $m + am'$ we deduce the existence of $\phi \in M^\vee$ with $\phi(m + am') = 1$. Then we can find an A-linear change of coordinates in $A \oplus M$ mapping (a, m) to $(0, m + am')$; indeed, composing the maps $(b, y) \mapsto (b, y + bm')$ and $(b, y) \mapsto (b - a\phi(y), y)$ we have

$$\begin{array}{ccccc} A \oplus M & \longrightarrow & A \oplus M & \longrightarrow & A \oplus M \\ (a, m) & \longmapsto & (a, m + am') & \longmapsto & (0, m + am'). \end{array}$$

Notice that this change of coordinates $\theta : A \oplus M \to A \oplus M$ further satisfies

$$\begin{array}{ccccc} A & \xrightarrow{h} & A \oplus M & \xrightarrow{\theta} & A \oplus M \\ 1 & \longmapsto & (a, m) & \longmapsto & (0, m + am'). \end{array}$$

Thus, denoting by C the cokernel of the split monomorphism $A \to M, 1 \mapsto m + am'$, we see that

$$N \cong \operatorname{Coker} h \cong \operatorname{Coker} \theta h \cong A \oplus C \cong M,$$

whence the theorem is implied by the lemma.

Let us continue with the proof. First we claim that we can reduce to the case when M projective. Indeed, we are given that $M \cong P \oplus V$ with P projective of free rank greater than d. Write $m = (p, v)$ with $p \in P, v \in V$. Since (a, p, v) is unimodular in $A \oplus M$, we see that $(\overline{a}, \overline{p})$ is unimodular in $\overline{A} \oplus \overline{P}$, where the bar denotes reduction modulo the order ideal $\mathfrak{o}(v)$.

As $\mathrm{rl}(\overline{P}) > d \geq \dim \overline{A}$,[26] the veracity of the lemma for projective modules (applied to $\overline{P}$) would imply the existence of $p' \in P$ such that $\overline{p} + \overline{ap'} \in \overline{P}$ is unimodular. Thus there is $\overline{\phi} : \overline{P} \to \overline{A}$ with $\overline{p} + \overline{ap'} \mapsto 1$. Using the projectivity of P we can lift $\overline{\phi}$ to a homomorphism $\phi : P \to A$. The commutativity of the diagram below implies the existence of $f \in V^\vee$ such that $\phi(p + ap') = 1 + f(v)$.

[26] *If* $\mathrm{rl}_A(P) = s$ *then when passing to the quotient a direct factor* A^s *in* P *becomes a direct factor* $\overline{A}^s$ *in* $\overline{P}$, *so that* $\mathrm{rl}_{\overline{A}}(\overline{P}) \geq \mathrm{rl}_A(P)$.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ \overline{P} \arrow[r, "\overline{\phi}"'] & \overline{A} \end{tikzcd}$$

Finally, setting $m' = (p', 0)$ it follows that the map $(\phi, -f) : P \oplus V \to A$ sends $m + am' = (p + ap', v)$ to $\phi(p + ap') - f(v) = 1$, whence $m + am'$ is unimodular, as desired.

Thanks to the lemma, we lose no generality in assuming that M is projective. Let us prove the claim by induction on $d = \dim \operatorname{Max} A$. The following result will be useful to deal with the case $d = 0$.

LEMMA 2.13: Let A be a semilocal ring,[27] $a \in A$ and $\mathfrak{b} \subseteq A$ an ideal satisfying $Aa + \mathfrak{b} = A$. Then there exists $y \in \mathfrak{b}$ such that $a + y \in A^\times$.

[27] *A (commutative) ring is* semilocal *if it has finitely many maximal ideals.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the maximal ideals of A. Denoting by $\mathfrak{J}$ the Jacobson radical, we have

$$A/\mathfrak{J} \cong A/\mathfrak{m}_1 \times \cdots \times A/\mathfrak{m}_n.$$

Thus the assertion reduces to verify the case of fields, which is evident. □

Suppose $d = 0$. As $\mathrm{rl}_A(M) > d$, Serre's theorem yields $M \cong A \oplus M_1$ for some A-module M_1. Write $m = (b, m_1)$ with $b \in A$, $m_1 \in M_1$, and $\mathfrak{i} = \{f(m_1) : f \in M_1^\vee\}$. Then the fact that (a, m) is unimodular implies the existence of $r, s \in A$, $\phi \in M_1^\vee$ such that $ra + sb + \phi(m_1) = 1$, or in other words

$$Aa + Ab + \mathfrak{i} = A.$$

But $d = 0$ implies that $\operatorname{Max} A$ is a finite set of points, so that A is semilocal and we are in a position to apply the previous lemma with respect to the ideal $\mathfrak{b} = Aa + \mathfrak{i}$. We obtain an element $a'a + f(m_1) \in \mathfrak{b}$ (with $a' \in A$, $f \in M_1^\vee$) such that $b + a'a + f(m_1) \in A^\times$. Consequently, setting $m' = (a', 0) \in A \oplus M_1$ one can rewrite this as $(\mathrm{id}, f)(m + a'm) \in A^\times$, whence $m + a'm$ is unimodular, as desired.

Next, assume that the claim holds for some $d - 1$ with $d \geq 1$. As before, Serre's theorem implies that $M \cong A \oplus M_1$ and we can write $m = (b, m_1)$ with $b \in A$, $m_1 \in M_1$. Let us deal with the case $\dim \operatorname{Max} A/bA \leq d - 1$ first. Reducing modulo b, it is clear that $\overline{a} + \overline{m_1}$ is unimodular in $\overline{A} \oplus \overline{M_1}$. As $M \cong A \oplus M_1$ implies that $\mathrm{rl}_{\overline{A}}(M_1) \geq d - 1$, we can apply the induction hypothesis to the $\overline{A}$-module $\overline{M_1}$ to deduce the existence of $m_1' \in M_1$ and $\overline{\phi} : \overline{M_1} \to \overline{A}$ satisfying

$$\overline{\psi}(\overline{m_1} + \overline{a}\overline{m_1'}) = 1.$$

Using the fact that M_1 is projective, we can lift $\overline{\psi}$ to an A-linear map $\psi : M_1 \to A$, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ \overline{M_1} \arrow[r, "\overline{\psi}"'] & \overline{A} \end{tikzcd}$$

From this we deduce the existence of $c \in A$ satisfying

$$\psi(m_1 + am_1') + cb = 1.$$

Hence letting $m' = (0, m_1') \in A \oplus M_1$, the map $(c, \psi) : A \oplus M \to A$, $(x, y) \mapsto cx + \psi(y)$ sends $m + am'$ to $cb + \psi(m_1 + am_1') = 1$, and we are done.

In the general case, by a change of coordinates we reduce the problem of the unimodularity of $(a, m) = (a, b, m_1) \in A \oplus A \oplus M_1$ to that of $(a, b + \alpha a + f(m_1), m_1) \in A \oplus A \oplus M_1$, with $\alpha \in A$, $f \in M_1^*$. Choose on each irreducible component pairwise distinct points x_i. Then we can find α and f such that $b + \alpha a + f(m_1) \in A_{x_i}^{\times}$, implying the desired conclusion $\dim \operatorname{Max} A/bA \leq d - 1$.

Given that (a, b, m_1) is unimodular in M_1, there exist $a', b' \in A$, $\phi \in M_1^*$ such that $aa' + bb' + \phi(m_1) = 1$. This allows us to replace b by $b + \lambda(1 - bb')$ in the above expresions, for $\lambda \in A$ an arbitrary constant. Finally, the Chinese remainder theorem we may choose λ so that $\lambda \equiv 0 \pmod{x_i}$ if $b \not\equiv 0 \pmod{x_i}$ and $\lambda \equiv 1 \pmod{x_i}$ if $b \equiv 0 \pmod{x_i}$, concluding the proof.[28] □

[28] *Parts of this proof may be rewritten using the notion of* semilocalization.

3 SOME FACTS FROM ALGEBRAIC GEOMETRY

In this section we introduce the notion of vector bundle in the algebraic setting. The constructions below, due to Grothendieck, will make use of some global versions of familiar algebraic objects: symmetric, tensor and alternating algebras, Spec and Proj. We have mostly followes [Gro61].

3.1 *Special sheaves of algebras*

Conventions: In this subsection, unless stated otherwise, rings are not assumed to be commutative; however, the symbol A will always denote a commutative ring. An *A-algebra* is a ring R together with a ring homomorphism $A \to R$ whose image is contained in the center of R; this is the same as R being an A-module such that scalar and ring multiplication are compatible. The categories of A-algebras and commutative A-algebras will be respectively denoted by A-**Alg** and A-$\mathbf{Alg}_c$.

Let us start by recalling the definition and basic properties of our objects of interest: the tensor and symmetric algebras.

DEFINITION 3.1: Let M be an A-module. The **tensor algebra** of M is defined as

$$T(M) = \bigoplus_{k>0} T^k(M)$$

where $T_0(M) = A$ and $T^k(M) = M^{\otimes k}$ for $k > 0$.

This is a ring whose multiplication is defined on generators as

$$(x_{i_1} \otimes \cdots \otimes x_{i_m}) \cdots (y_{j_1} \otimes \cdot \otimes y_{j_n}) = x_{i_1} \otimes \cdots \otimes x_{i_m} \otimes y_{j_1} \otimes \cdots \otimes y_{j_n}$$

and then extending A-linearly. The canonical injection $A = T^0(M) \to T(M)$ gives $T(M)$ the structure of an A-algebra; in addition, the direct sum decomposition in its definition is readily seen to be a grading. Notice that we have a canonical injection $i : M = T^1(M) \to T(M)$.

Next we state the universal property of the tensor algebra; in some sense, it is the "smallest" A-algebra containing M.

PROPOSITION 3.2: Let M be an A-module. For every A-algebra E and every A-linear map $f : M \to E$ there exists a unique A-algebra homomorphism g such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & T(M) \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

A formal reformulation of this fact is the following:

PROPOSITION 3.3: The functor $T : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Alg}, M \mapsto T(M)$ is left adjoint to the forgetful functor $A\text{-}\mathbf{Alg} \to A\text{-}\mathbf{Mod}$.[29]

Now we turn to our next definition.[30]

[29] *Regarding $T(M)$ as a graded A-algebra, it turns out that $T : A\text{-}\mathbf{Mod} \to \mathbf{Gr}\ A\text{-}\mathbf{Alg}, M \mapsto T(M)$ is left adjoint to $\mathbf{Gr}A\text{-}\mathbf{Alg} \to A\text{-}\mathbf{Mod}, E = \bigoplus_{i \geq 0} E_i \mapsto E_1$.*

[30] *Cf. EGA II.1.7, p. 14.*

DEFINITION 3.4: Let I be the (two-sided) ideal of $T(M)$ generated by all the elements of the form $x \otimes y - y \otimes x$, $x, y \in M$. The **symmetric A-algebra** of M is defined as the quotient $S(M) = T(M)/I$.

The quotient $T(M)/I$ inherits a graded A-algebra structure from $T(M)$. It must be pointed out that, unlike $T(M)$, this algebra is *commutative* (of course, this is the purpose of the construction). Let us denote by $\pi : M \to S(M)$ the canonical projection.

We can also characterize the symmetric algebra by means of a universal property: it is the "smallest" *commutative* A-algebra containing M.

PROPOSITION 3.5: For every A-linear map f from M into a commutative A-algebra B, there is a unique A-algebra homomorphism $g : S(M) \to B$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "f"'] & S(M) \arrow[dl, dashed, "g"] \\ B & \end{tikzcd}$$

In categorical language, this amounts to saying that:

PROPOSITION 3.6: The functor $S : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Alg}_c, M \mapsto S(M)$ is left adjoint to the forgetful functor $A\text{-}\mathbf{Alg}_c \to A\text{-}\mathbf{Mod}$.

We also point out some simple properties concerning S:

- $S(M \oplus N) \cong S(M) \oplus_A S(N)$ canonically.
- S commutes with $\varinjlim$.

EXAMPLE 3.7: For $n > 0$, the symmetric algebra $S(A^n)$ is isomorphic to the polynomial ring $A[x_1, \ldots, x_n]$. For this reason, this particular example of the symmetric algebra is sometimes said to be akin to a polynomial ring "without choosing coordinates".

3.2 *Relative spectrum of a quasicoherent sheaf*

We now discuss a "global" version of the prime spectrum of a ring.

First, recall that a morphism $f : X \to S$ is *affine* if there is an affine open cover $S = \bigcup S_i$ such that each $f^{-1}(S_i)$ is affine. An important property is the following:

PROPOSITION 3.8: Let $f : X \to S$ be affine. Then for every open subset U of S, the inverse image $f^{-1}(U)$ is affine.

THEOREM 3.9: Let S be a scheme and $\mathscr{B}$ be a quasicoherent $\mathscr{O}_S$-algebra. There is an affine S-scheme X, unique up to S-isomorphism, such that $\mathscr{B}$ and $f_*\mathscr{O}_X$ are isomorphic as $\mathscr{O}_S$-algebras. This is called the **relative spectrum** of $\mathscr{B}$ and we shall denote it by $\mathrm{Spec}(\mathscr{B})$.

Some properties:

- The canonical morphism $\pi : \mathrm{Spec}(\mathscr{B}) \to S$ is *affine*. In particular, the inverse image $\pi^{-1}(U)$ of an open subset U of S is affine with ring $\Gamma(U, \pi * \mathscr{O}_X)$.
- There is a canonical isomorphism of $\mathscr{O}_S$-algebras $\pi_*\mathscr{O}_{\mathrm{Spec}(\mathscr{B})} \cong \mathscr{B}$.
- For every affine morphism $Y \xrightarrow{\alpha} S$, there is a canonical isomorphism $\mathrm{Spec}(\alpha_*\mathscr{O}_Y) \cong Y$ as S-schemes.

We point out that one can characterize Spec by means of a universal property.

PROPOSITION 3.10: $\mathrm{Spec}(\mathscr{B})$ represents the functor $(S\text{-}\mathbf{Sch})^{op} \to \mathbf{Set}$ mapping an S-scheme $Y \xrightarrow{\alpha} S$ to the set of all possible $\mathscr{O}_S$-algebra morphisms $\mathscr{B} \to \alpha_*\mathscr{O}_Y$.

Next, we pass to a global version of symmetric algebras.

DEFINITION 3.11: Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. The **symmetric algebra** of $\mathscr{F}$ is the sheaf $S(\mathscr{F})$ on X associated to the presheaf of A-algebras $U \mapsto S(\mathscr{O}_X(U))$.

We remark that the symmetric algebra of a sheaf has a universal property analogous to that of the symmetric algebra of a module, though we will not repeat the statement for being completely analogous.

We summarize below some simple properties of the symmetric algebra sheaf:

- What are stalks.
- For all $\mathscr{E}$ and $\mathscr{F}$ we have $S(\mathscr{E} \oplus \mathscr{F}) \cong S(\mathscr{E}) \otimes S(\mathscr{F})$.
- Let $\mathscr{L}$ be an invertible sheaf. Then we have
$$S(\mathscr{L}) = \bigoplus_{n \geq 0} \mathscr{L}^n.$$

EXAMPLE 3.12: For the structure sheaf $\mathscr{F}$, the symmetric algebra $S(\mathscr{F})$ is isomorphic to the *polynomial sheaf* $\mathscr{F}[T] = \mathscr{F} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$.[31] In general, for the constant sheaf $\mathscr{F}^n$ we have an isomorphism $S(\mathscr{F}^n) \cong \mathscr{F}[T_1, \ldots, T_n]$.

[31] *Here $\mathbb{Z}$ is regarded as a constant sheaf.*

The next propositions concern the behavior of the symmetric algebra of quasicoherent sheaves.

PROPOSITION 3.13: Let X be a scheme and $\mathscr{F}$ be a quasicoherent (resp. of finite type) sheaf on X. Then $S(\mathscr{F})$ is quasicoherent (resp. of finite type).[32]

[32] *The same holds for $T(\mathscr{F})$ but I don't need it right now.*

PROPOSITION 3.14: For an A-module M we have that $S(\tilde{M}) = \widetilde{S(M)}$.

3.3 *Affine vector bundles*

We are now ready to discuss the notion of vector bundle in the algebro-geometric setting.

DEFINITION 3.15: An *algebraic vector bundle of rank n* on a scheme X is a locally free $\mathscr{O}_X$-module of constant finite rank n.

A more familiar definition, closer in spirit to the original, is the following: a *geometric vector bundle of rank n* on X is a scheme $f : Y \to X$ together with an open covering $Y = \bigcup Y_i$ and isomorphisms $\phi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ such that for all i, j and for every affine open subset $V = \mathrm{Spec}\, A$ contained in $U_i \cap U_j$, the automorphism $\phi_j \phi_i^{-1}$ of $\mathbb{A}^n_V = A[T_1, \ldots, T_n]$ is *A-linear*.

We can relate both definitions by means of the following construction.

DEFINITION 3.16: Let X be a scheme and $\mathscr{F}$ be a quasicoherent $\mathscr{O}_X$-module. The **algebraic vector bundle associated to** $\mathscr{F}$ is defined as the spectrum of the quasicoherent $\mathscr{O}_X$-algebra $S(\mathscr{F})$. Notation: $V(\mathscr{F})$.

PROPOSITION 3.17: Let X be a scheme. There is an equivalence of categories between the category of algebraic vector bundles of rank n and the category of geometric vector bundles of rank n.

Proof. We will briefly describe the bijection without delving into details. Given a locally free sheaf of rank n over X, the previously defined $V(\mathscr{F})$ turns out to be a geometric vector bundle of rank n. On the other hand, given a geometric vector bundle $Y \to X$ of rank n, we can naturally construct its sheaf of sections; this turns out to have a natural $\mathscr{O}_X$-module structure, which implies that it is locally free of rank n. □

In fact, there is another notable correspondence, first noticed by Serre in [Ser58], regarding algebraic vector bundles over an affine scheme. The full statement below is sometimes referred as the *Swan-Serre theorem*.[33]

[33] The contribution of the American mathematician Richard Swan concerns a similar equivalence between vector bundles on a compact Hausdorff topological space X and finitely generated projective modules over the ring of continuous functions on X.

THEOREM 3.18 (SERRE-SWAN): Let A be a ring. There is an equivalence of categories between the category of algebraic vector bundles of rank n over $\operatorname{Spec} A$ and finitely generated projective A-modules.

Proof. The equivalence is given as follows. Given an algebraic vector bundle $\mathscr{F}$ of rank n over $X = \operatorname{Spec} A$, its global sections $\Gamma(X, \mathscr{F})$ turn out to be a finitely generated projective module of rank n over $\Gamma(X, \mathscr{O}_X) = A$. Conversely, given a finitely generated projective A-module of rank n, taking the associated quasicoherent sheaf $\widetilde{M}$, it can be immediately verified that it is locally free of constant rank n. □

3.4 *Relative homogeneous spectrum of a quasicoherent sheaf*

In this section we will review the concept of "relative Proj" for a sheaf. We start by briefly recalling the classical Proj construction.

Let B be a ring. A *grading* is a decomposition $B = \bigoplus_{d \geq 0} B_d$ where B_d are abelian subgroups and $B_d B_e \subseteq B_{d+e}$ for all d, e. We say that B is a *graded ring* and elements of B_d are called *homogeneous elements of degree d*.[34] If in addition B is an A-algebra for some ring A, we say that B is a *graded A-algebra* if the image of A is contained in B_0. (Notice that the grading condition implies that B_0 is a subring of B.) In a similar fashion, a *grading* of a B-module M is a decomposition $M = \bigoplus_{d \geq 0} M_d$ where M_d are abelian subgroups and $B_d M_e \subseteq M_{d+e}$ for all d, e.

[34] *In general an element of B is uniquely expressed as a sum of homogeneous elements, its* homogeneous parts.

EXAMPLE 3.19: $A[x] = \bigoplus_{d \geq 0} A x^d$. In more variables the grading is given by the subgroups generated by monomials of the same degree.

Let B be a graded A-algebra. Recall that an ideal $I \subseteq B$ is *homogeneous* if it is generated by homogeneous elements.[35,36]

[35] *Equivalently,* $I = \bigoplus_{d \geq 0} (I \cap B_d)$, *which is the same as saying that for all* $x \in I$, *each of the homogeneous parts of* x *belongs to* I *as well.*

[36] *Nice fact: if* I *is homogeneous then* B/I *has a natural grading given by* $B/I = \bigoplus B_d/(I \cap B_d)$.

DEFINITION 3.20: For a graded A-algebra B, the set $\operatorname{Proj} B$ is the collection of all homogeneous prime ideals of B not containing the *irrelevant ideal* $B_+ = \bigoplus_{d>0} B_d$. The *Zariski topology* on $\operatorname{Proj} B$ is defined by declaring the sets $V_+(I) = \{\mathfrak{p} \in \operatorname{Proj} B : \mathfrak{p} \supseteq I\}$ to be closed, where I is an arbitrary homogenous ideal of B.

The sets of the form $D_+(f) = \operatorname{Proj} B \backslash V_+(fB)$, with $f \in B$ homogeneous, forms a basis for the Zariski topology on B; these are called *principal open sets*. It is well-known that $X = \operatorname{Proj} B$ can be endowed with a unique A-scheme structure $(X, \mathscr{O}_X)$ in such a way that for every homogeneous $f \in B$ there is a canonical isomorphism $D_+(f) \cong \operatorname{Spec} B_{(f)}$ of A-schemes.[37]

[37] *Here* $B_{(f)} = \{af^{-n} \in B_f : n \geq 0, \deg a = n \deg f\}$. *This is nothing but the subring of* B_f *formed by the elements of "degree zero", and is sometimes called* homogeneous localization.

The following result concerns the "functoriality" of Proj.

PROPOSITION 3.21: Let $\phi : C \to B$ be a homomorphism of graded rings, and M be the homogeneous ideal $\phi(C_+)B \subseteq B$. Then ϕ induces a morphism

$$\phi^{\#} : \operatorname{Proj} B \backslash V_+(M) \to \operatorname{Proj} C$$

such that for every homogeneous $h \in C_+$ one has $f^{-1}(D_+(h))$ and furthermore $f|_{D_+(h)}$ coincides with the morphism of schemes induced by $C_{(h)} \to B_{(h)}$.

Notice that in general a map $C \to B$ does not induce a morphism $\operatorname{Proj} B \to \operatorname{Proj} C$; however, an important case for which this hold is that of *surjective* maps $C \twoheadrightarrow B$.

Now we "globalize" some of the concepts above.[38]

[38] *Cf.* [Gro61], §3.1, p. 49.

DEFINITION 3.22: Let Y be a scheme. A *graded $\mathscr{O}_Y$-algebra* is a sheaf $\mathscr{S}$ admitting a decomposition $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{S}_d$ for some $\mathscr{O}_Y$-modules $\mathscr{S}_d$, such that $\mathscr{S}_0 = \mathscr{O}_Y$ and for every open set $U \subseteq Y$ the decomposition $\mathscr{S}(U) = \bigoplus_{d \geq 0} \mathscr{S}_d(U)$ is a grading of the $\mathscr{O}_Y(U)$-algebra $\mathscr{S}(U)$. A *graded $\mathscr{S}$-module* is an $\mathscr{S}$-module $\mathscr{M}$ admitting a decomposition $\mathscr{M} = \bigoplus_{d \geq 0} \mathscr{M}_d$ for some $\mathscr{S}$-modules $\mathscr{M}_d$, such that for every open set $U \subseteq X$, the decomposition $\mathscr{M}(U) = \bigoplus_{d \geq 0} \mathscr{M}_d(U)$ is a grading of the $\mathscr{S}(U)$-algebra $\mathscr{M}(U)$.

DEFINITION 3.23: Let Y be a scheme. For every quasicoherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ there exists a Y-scheme $X \xrightarrow{f} Y$, unique up to Y-isomorphism, such that:

- For every affine open $U \subseteq Y$ there is an isomorphism $\eta_U : f^{-1}(U) \xrightarrow{\sim} \operatorname{Proj} \mathscr{S}(U)$.
- For every two affine open subsets $V \subseteq U \subseteq Y$, there is a commutative diagram

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "\eta_V", "\sim"'] \arrow[d] & \operatorname{Proj} \mathscr{S}(V) \arrow[d] \\ f^{-1}(U) \arrow[r, "\sim", "\eta_U"'] & \operatorname{Proj} \mathscr{S}(U) \end{tikzcd}$$

  The Y-scheme X is called the *relative homogeneous spectrum* of $\mathscr{S}$ and we will denote it by $\operatorname{Proj} \mathscr{S}$.

Notice that for every open subscheme $U \subseteq Y$, the U-scheme $f^{-1}(U)$ is identified with $\operatorname{Proj} \mathscr{S}|_U$.

3.5 *Projective vector bundles*

Next we introduce the notion of *projective bundles*, which plainly speaking are vector bundles (in the usual geometric sense) with projective spaces as fibers.

DEFINITION 3.24: Let Y be a scheme, $\mathscr{F}$ a quasicoherent $\mathscr{O}_Y$ module. The *projective bundle over Y defined by $\mathscr{F}$* is the Y-scheme $P = \operatorname{Proj} S_{\mathscr{O}_Y}(\mathscr{F})$. The $\mathscr{O}_P$-module $\mathscr{O}_P(1)$ is called the *fundamental sheaf* on P.[39]

[39] *Warning: some autors insist that the terminology "bundle" should be used only for locally free sheaves. This may not be the case in this definition.*

Due to the correspondence between finitely generated projective modules and vector bundles, we may look at *sections* of a bundle in order to extract information about direct summands of rank 1. This is essential in the proof of the final result.

More generally, we can describe morphisms into a projective bundle; this is the goal of this paragraph. As before, let Y be a scheme, $\mathscr{F}$ a quasicoherent $\mathscr{O}_Y$-module and $P = \mathbb{P}(\mathscr{F})$; call p the structure morphism $P \to Y$. We wish to classify Y-morphisms into P. The main result is the following.

THEOREM 3.25: Let $q : X \to Y$ be a morphism of schemes. Consider the collection Σ of pairs $(\mathscr{L}, \phi)$ with $\mathscr{L}$ an invertible $\mathscr{O}_X$-module and $\phi : q^*\mathscr{F} \to \mathscr{L}$ a surjective $\mathscr{O}_X$-homomorphism. Introduce an equivalence relation $\sim$ as follows: $(\mathscr{L}, \phi) \sim (\mathscr{L}', \phi')$ if and only if there is an $\mathscr{O}_X$-isomorphism $\tau : \mathscr{L} \to \mathscr{L}'$ such that $\phi' = \tau\phi$. Finally let $\Xi = \Sigma / \sim$. Then there is a one-to-one correspondence between $\mathbf{Sch}_Y(X, \mathbb{P}(\mathscr{F}))$ and Ξ.

Proof. Let $r \in \mathbf{Sch}_Y(X, \mathbb{P}(\mathscr{F}))$. Start with the following observation. We know that the inclusion $\mathscr{F} \hookrightarrow S(\mathscr{F})$ induces a homomorphism $\alpha : \mathscr{F} \to$

$p_*\mathscr{O}_P(1)$, so by the adjunction between direct and inverse image we obtain a homomorphism $\alpha^\# : p^*\mathscr{F} \to \mathscr{O}_P(1)$.

LEMMA 3.26: The homomorphism $\alpha^\# : p^*\mathscr{F} \to \mathscr{O}_P(1)$ is surjective.

Then $\phi_r = r^*\alpha^\# : r^*p^*\mathscr{F} \to r^*\mathscr{O}_P(1)$ is also surjective.[40] Here notice that $pr = q$ implies $r^*p^*\mathscr{F} = q^*\mathscr{F}$; on the other hand $\mathscr{L}_r = r^*\mathscr{O}_P(1)$ is an invertible $\mathscr{O}_X$-module.[41] Consequently we can assign to r a pair $(\mathscr{L}_r, \phi_r)$ with $\mathscr{L}_r$ an invertible sheaf on X and $\phi_r : q^*\mathscr{F} \to \mathscr{L}_r$ a surjective $\mathscr{O}_X$-homomorphism.[42] Conversely, let $\mathscr{L}$ be an invertible sheaf on X and $\phi : q^*\mathscr{F} \to \mathscr{L}$ an $\mathscr{O}_X$-homomorphism. Then ϕ induces an $\mathscr{O}_X$-homomorphism

$$\psi : q^*S(\mathscr{F}) \to \bigoplus_{n\geq 0} \mathscr{L}^n$$

[43]and since ϕ is surjective, we have an induced Y-morphism

$$r_{\mathscr{L},\phi} : X \to \mathbb{P}(\mathscr{F}).$$

Thus we have the following correspondence:

$$\begin{array}{ccc} \mathbf{Sch}_Y(X, \mathbb{P}(\mathscr{F})) & \longleftrightarrow & \Xi \\ r & \longmapsto & (\mathscr{L}_r, \phi_r) \\ r_{\mathscr{L},\phi} & \longleftarrow\!\shortmid & (\mathscr{L}, \phi) \end{array}$$

[44]In order to prove this is a bijection, we need to study in more detail morphisms of the form $q^*\mathscr{S} \to \bigoplus_{n\geq 0} \mathscr{L}^n$, where $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra and $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module.

By the adjunction between direct image and pullback, a morphism $\psi : q^*\mathscr{S} \to \bigoplus_{n\geq 0} \mathscr{L}^n$ corresponds to a morphism $\psi^\flat : \mathscr{S} \to \bigoplus_{n\geq 0} \mathscr{L}^n$. Recalling that $\operatorname{Proj}(\bigoplus_{n\geq 0} \mathscr{L}^n) \cong X$, we see that $\psi^\flat$ induces a morphism $r_{\mathscr{L},\psi} : G_\psi \to \operatorname{Proj}\mathscr{S}$ for some open subset G_ψ of X. Notice that such morphism can be expressed as follows:

$$\begin{tikzcd} G_\psi \arrow[rr, "r"] \arrow[dr, "\tau"'] & & \operatorname{Proj}\mathscr{S} \\ & \operatorname{Proj} q^*\mathscr{S} \arrow[ur, "\pi"'] & \end{tikzcd}$$

Here τ is the morphism induced by ψ, while π is the canonical projection $\operatorname{Proj} q^*\mathscr{S} \cong \operatorname{Proj}\mathscr{S} \times_Y X \to \operatorname{Proj}\mathscr{S}$.

Assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, so that $\mathscr{S} = \widetilde{S}$ and $\mathscr{L} = \widetilde{L}$ with S an A-algebra and L a free B-module of rank 1. Furthermore $q^*\mathscr{S}(S \otimes_A B)^\sim$. Taking a generator c for L, for each n the morphism $q^*\mathscr{S}_n \to \mathscr{L}^{\otimes n}$ corresponds to an A-linear map

$$\begin{aligned} S_n \otimes_A B &\longrightarrow L^{\otimes n} \\ s \otimes b &\longmapsto b v_n(s) c^{\otimes n} \end{aligned} \tag{1}$$

for some A-linear map $v_n : S_n \to B$; in fact, the collection of maps $\{v_n\}_{n\geq 0}$ defines a homomorphism of A-algebras $S \to B$.

On the other hand, letting $f \in S_d$ for some $d > 0$ and $g = v_d(f)$, we have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ and $\tau^{-1}(D_+(f \otimes 1)) = D(g)$ by the very definition of π and τ, so that $r^{-1}(D_+(f)) = D(g)$. Restricting to $D(g)$ and $D_+(f \otimes 1)$ we see that τ and π have the form

$$\begin{array}{ccc} D(g) & \xrightarrow{\tau} & \operatorname{Proj} q^*\mathscr{S} \\ \dfrac{s\otimes 1}{(f\otimes 1)^n} & \longmapsto & \dfrac{v_{nd}(s)}{g^n} \end{array} \qquad\qquad \begin{array}{ccc} D_+(f\otimes 1) & \xrightarrow{\pi} & \operatorname{Proj}\mathscr{S} \\ \dfrac{s}{f^n} & \longmapsto & \dfrac{s\otimes 1}{(f\otimes 1)^n} \end{array}$$

whence the restriction of r to $D(g)$ corresponds to a ring morphism

$$\begin{array}{ccc} S_{(f)} & \xrightarrow{r} & B_g \\ \dfrac{s}{f^n} & \longmapsto & \dfrac{v_{nd}(s)}{g^n} \end{array} \tag{2}$$

[40] *The inverse image functor is right exact, being left adjoint to the direct image functor.*

[41] *The inverse image of an invertible sheaf is an invertible sheaf.*

[42] *Keep in mind that S commutes with inverse images and that S of an invertible sheaf is just the right hand side.*

[43] *Cf. [Gro61], 3.7.1.*

[44] *Cf. [Gro61], 3.7.2.*

We will need a more explicit description of $\phi_r : q^*\mathscr{F} \to \mathscr{L}_r$. Let $Y = \operatorname{Spec} A$ be affine, so that $\mathscr{F} = \widetilde{F}$ for some A-module F. By the adjunction between direct image and pullback, we have a morphism $\phi_r^\flat : \mathscr{F} \to q_*\mathscr{L}_r$. Using this we can write for every $f \in S_d$ (with $d > 0$)

$$r^{-1}(D_+(f)) = X_{\phi_r^\flat(f)}.$$

Now let $V = \operatorname{Spec} B$ be an affine open set contained in $r^{-1}(D_+(f))$; thus B is an A-algebra. We also have $q^*\mathscr{F}|_V = (F \otimes_A B)^\sim$ and $\mathscr{L}_r|_V = \widetilde{L_r}$, where $L_r = S(1) \otimes_{S_{(f)}} B$. Letting $S = S_A(F)$, we see that $r : V \to D_+(f)$ corresponds to an A-morphism $\omega : S_{(f)} \to B$, and the restricted map $\phi_r : q^*\mathscr{F}|_V \to \mathscr{L}_r$ corresponds to a morphism of B-algebras

$$\begin{aligned} E \otimes_A B &\longrightarrow L_r \\ x \otimes 1 &\longmapsto \frac{x}{1} \otimes f \end{aligned}$$

Here notice that $x/1 \otimes f = f/1 \otimes \omega(x/f)$.

It follows that the canonical extension of ϕ_r into a morphism of $\mathscr{O}_X$-algebras

$$q^*S(\mathscr{F}) \xrightarrow{\psi_r} \bigoplus_{n \geq 0} \mathscr{L}_r^{\otimes n}$$

such that, for each n, the restricted morphism $\psi_r : q^*S_n(\mathscr{F})|_V \to S_n(\mathscr{L}_r)$ corresponds to a morphism

$$\begin{aligned} S_n(F) \otimes_A B &\longrightarrow L_r^{\otimes n} \\ s \otimes 1 &\longmapsto \left(\frac{f}{1}\right)^n \otimes \omega\left(\frac{s}{f^n}\right). \end{aligned}$$

[45]We are finally ready to tackle the main theorem. Let a Y-morphism $R : X \to \mathbb{P}(\mathscr{F})$ be given, and construct $\mathscr{L}_r$ and ϕ_r; set also $r' = r_{\mathscr{L}_r, \phi_r}$ and take $f \in S_d$, $V = \operatorname{Spec} B \subseteq r^{-1}(D_+(f))$ as before. Notice that L_r is generated by $f/1 \otimes 1$, so that for each n the morphism v_n in 1 takes the form $v_n : s \mapsto \omega(s/f^n)$. Thus by the computation made in 2 we have

[45] *Correct this paragraph (the writing is terrible).*

$$r' : \frac{s}{f^n} \mapsto \frac{v_{nd}(s)}{v_d(f)^n} = \frac{\omega(s/f^{nd})}{\omega(f^n/f^{nd})} = \omega\left(\frac{s}{f^n}\right).$$

Therefore $r' = r$. The converse of the statement is immediate and will be omitted.

□

4 A SPLITTING THEOREM FOR PROJECTIVE MODULES

In the last section of this document we discuss the existence of certain splitting property for projective modules. This result due to Gabber, Liu and Lorenzini; cf. [GLL15]. First, their result on the existence of hypersurfaces is discussed. As a particular application, the existence of finite quasi-sections for certain morphisms is established. As a final application, the claimed splitting property is proved.

4.1 *Existence of hypersurfaces*

Hypersurfaces in the classical sense are varieties which are the zero locus of a single polynomial equation in affine (or projective) space. We will consider a more general definition of hypersurface relative to a morphism.

We start by making sense of the "zero locus" of a global section of an invertible sheaf.

LEMMA 4.1: [46] Let X be a locally ringed space, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, $f \in \mathscr{L}(X)$, and $x \in X$. The following conditions are equivalent:

[46] *EGA 0, §5.5.2, p. 53.*

(a) f_x is a generator of $\mathscr{L}_x$.

(b) $f_x \notin \mathfrak{m}_x \mathscr{L}_x$ (i.e. $f(x) \neq 0$).[47]

[47] Here $f(x)$ is the class of $f_x \mod \mathfrak{m}_x$.

(c) There is a section $g \in \mathscr{L}^{-1}(V)$ for some open neighborhood V of x such that the canonical image of $f \otimes g$ in $\mathscr{O}_X(V)$ is the unit section.[48]

[48] Remind yourself (5.4.3) that there is a canonical isomorphism $\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X$.

Proof. The question is local so one may assume $\mathscr{L} = \mathscr{O}_X$. Then (a) $\Leftrightarrow$ (b) is obvious. If (b) holds, then $f_x \in \mathscr{O}_X^*$ so that $f_x g_x = 1$ for some section $g \in \mathscr{O}_x(V)$ defined on a neighborhood V of x. It follows that $fg = 1$ on V, from which (c) is clear. $\square$

Let us define the subset $X_f = \{x \in X : f(x) \neq 0\}$ of X; part (c) above implies that it is open. Thus the "zero locus" $H_f = X \backslash X_f$ is a closed subset of X. We endow it with a closed subscheme structure via the sheaf of ideals $\mathscr{O}_X f \otimes \mathscr{L}^{-1}$.

EXAMPLE 4.2: $X = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}(d)$.

DEFINITION 4.3: Let S be an affine scheme, $X \to S$ a morphism and $\mathscr{L}, f$ as above. We say that H_f is a *hypersurface relative to* $X \to S$ if no positive-dimensional irreducible component of X_s is contained in H_f for all $s \in S$. In addition, if the sheaf $\mathscr{I}$ of H_f is invertible, we say that H_f is *locally principal*.

EXAMPLE 4.4: Let $X \to \operatorname{Spec} A$ be a projective morphism, so that $X = \operatorname{Proj} B$ with B being a finitely generated homogeneous algebra over A. The embedding of X in projective space corresponds to a very ample sheaf $\mathscr{O}_X(1)$. The global sections $\Gamma(X, \mathscr{O}_X(n))$ may be identified with the degree n homogeneous elements of B. It follows that $H_f = V_+(f) \subseteq X$ and hypersurfaces as defined above coincide with the usual notion.

We point out some elementary properties of hypersurfaces.

LEMMA 4.5: Let S be an affine scheme, and $X \to S$ an affine morphism. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $f \in \Gamma(X, \mathscr{L})$ be such that $H = H_f$ is a hypersurface relative to $X \to S$.

(1) For $s \in S$ such that $\dim X_s \geq 1$, we have $\dim H_s \leq \dim X_s - 1$. Furthermore, if $X \to S$ is projective, $\mathscr{L}$ is ample and $H \neq \emptyset$, then H_s meets every positive-dimensional irreducible component of X_s, and consequently $\dim H_s = \dim X_s - 1$.

(2) H is finitely presented over S.

Proof.

(1) If $H_s = \emptyset$ then $\dim H_s < 0$ by convention, and the claimed inequality holds. Thus we can assume $H_s \neq \emptyset$.

(2) H is locally define as the zero set of a single equation in X, whence finite presentation is clear. $\square$

Following [GLL15], the main result is the following.

THEOREM 4.6: Let S be a finite-dimensional affine Noetherian scheme. Let $X \to S$ be a quasi-projective morphism and $\mathscr{O}_X(1)$ be the corresponding very ample sheaf on X. Suppose the following are given:

(i) A closed subscheme C of X.

(ii) A finite collection $F_1, \ldots F_m$ of locally closed subsets[49] of X, such that for all $s \in S$ and $i = 1, \ldots, m$ no positive-dimensional irreducible component of $F_{i,s}$ is contained in C_s.

[49] Recall that a subset F of a topological space X is locally closed *if it is the intersection of an open subset $U \subseteq X$ with a closed subset $Z \subseteq X$. In the case when X is a scheme, F can be given a subscheme structure by regarding U as an open subscheme of X, and $F = Z \cap U$ as a closed subscheme of U (with the reduced structure).*

(iii) A finite subset A of X with $A \cap C = \emptyset$.

Then there exists $n_0 > 0$ such that for all $n \geq n_0$ there is $f \in \Gamma(X, \mathscr{O}_X(n))$ satisfying:

1. C is a closed subscheme of H_f.

2. For all $s \in S$ and $i = 1, \dots, m$ no positive-dimensional irreducible component of $F_i \cap X_s$ is contained in H_f.

3. $H_f \cap A = \emptyset$.

Furthermore, if for all $s \in S$ no positive-dimensional irreducible component of X_s is contained in C, then f may be chosen so that H_f is a hypersurface relative to $X \to S$.

The proof of this result is rather elabore and a comprehensive exposition will not be given.[50] In order to provide the reader with a gist of the argument, below we briefly explain the underlying ideas.

[50] *For details, please see [GLL15], Theorem 3.3.*

Let $\mathscr{I}$ be the sheaf defining the closed subscheme and $\mathscr{I}(n) = \mathscr{I} \otimes \mathscr{O}_X(n)$. The goal of the argument is to establish that, for n large enough, there is a global section f of $\mathscr{I}(n)$ such that the corresponding H_f has the claimed properties. In order to achieve this, fix a set of generators $f_1, \dots, f_N \in \Gamma(X, \mathscr{I}(n))$ and for each $s \in S$ consider the subset $\Sigma_s \subset \mathbb{A}^N(k(s))$ consisting of all the $(\alpha_1, \dots, \alpha_N)$ such that $\sum \alpha_i f_i|_{X_s}$ has the claimed properties. It is then established that all such subsets Σ_s are the rational points of a single *pro-constructible*[51] subset T of $\mathbb{A}^N_S$ (of course depending on n). In order to find a global section $f = \sum \alpha_i f_i$ ($\alpha_i \in A$) that avoids T, we need to recur to the following technical result:

[51] *Recall that a subset $T \subseteq X$ is* constructible *if it is a finite union of sets of the form $U \cap (X \backslash V)$ with $U, V \subseteq X$ retrocompact. We say it is* locally constructible *if every point $t \in T$ has an open neighborhood $V \subseteq X$ such that $T \cap V$ is constructible in V. Finally , a subset of a Noetherian scheme X is called* pro-constructible *if it is a (possibly infinite) intersection of constructible subsets of X.*

THEOREM 4.7: Let S be a Noetherian affine scheme and $T = T_1 \cup \cdots \cup T_m$ be a finite union of pro-constructible subsets of $\mathbb{A}^N_S$. Assume that:

(1) For all $i = 1, \dots, m$ one has $\dim T_i < N$ and $T_{i,s}$ is constructible in $\mathbb{A}^N_{k(s)}$ for all $s \in S$.

(2) For all $s \in S$ there is a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ not belonging to T_s.

Then there exists a section σ of $\mathbb{A}^N_S \to S$ with $\sigma(S) \cap T = \emptyset$.

Then it is shown that T satisfies the hypotheses of the previous statement for large n, which establishes the existence of a suitable vector $(\alpha_1, \dots, \alpha_N) \in A^N$, which defines the desired global section $f = \sum \alpha_i f_i$. The interested reader may find the details in the first two sections of [GLL15].

It is possible to generalize the previous theorem to the case when S is not necessarily Noetherian. This is the version of the theorem that we will use in the final result.

THEOREM 4.8: Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism, with $\mathscr{O}_X(1)$ the corresponding very ample sheaf on X. Suppose the following are given:

(i) A closed subscheme C of X, finitely presented over S.

(ii) A finite collection $F_1, \dots F_m$ of subschemes of X, such that for all $s \in S$ and $i = 1, \dots, m$ no positive-dimensional irreducible component of $F_{i,s}$ and of X_s is contained in C.

(iii) A finite subset A of X with $A \cap C = \emptyset$.

Assume that for all $s \in S$ no positive-dimensional irreducible component of X_s is contained in C. Then there exists $n_0 > 0$ such that for all $n \geq n_0$ there is $f \in \Gamma(X, \mathscr{O}_X(n))$ satisfying:

1. The closed subscheme H_f of X is a hypersurface containing C as a closed subscheme.

2. For all $s \in S$ and $i = 1, \dots, m$ no positive-dimensional irreducible component of $F_{i,s}$ is contained in H_f.

3. $H_f \cap A = \emptyset$.

Proof. [GLL15] Theorem 5.1

First notice that by adding X to the collection $F_1, \ldots, F_m$, the condition that H_f be a hypersurface is precisely part (2) of the conclusion.

The proof essentially consists in reducing to the case when S is finite-dimensional Noetherian; this is a standard application of arguments for "removing Noetherian hypotheses" in algebraic geometry. Using [Gro66], Propositions 8.9.1 and 8.10.5, the data of the theorem "descends" as follows: there exist an affine scheme of finite type over $\mathbb{Z}$, a ring homomorphism $S \to S_0$ and a quasi-projective scheme $X_0 \to S_0$ such that $X \cong X_0 \times_{S_0} S$. Furthermore, the sheaf $\mathscr{O}_X(1)$ also "descends": there is a very ample sheaf $\mathscr{O}_{X_0}(1)$ relative to $X_0 \to S_0$ whose pullback along the canonical projection $p : X \to X_0$ is $\mathscr{O}_X(1)$. The same happens with the other objects: there is a closed subscheme C_0 of X_0 such that $C \cong C_0 \times_{S_0} S$, and finally there are subschemes $F_{1,0}, \ldots, F_{m,0}$ of X_0 such that $F_i \cong F_{i,0} \times_{S_0} S$. Being of finite type over $\mathbb{Z}$, we know that S_0 is finite-dimensional Noetherian, and the descent procedure will allow us to perform the desired reduction.

Call $A_0 = p(A)$. Consider the data X_0, C_0, $F_{1,0} \backslash C_0, \ldots, F_{m,0} \backslash C_0$, A_0. Then we can apply Theorem 4.6 to deduce the existence of $n_0 > 0$ and $f_0 \in \Gamma(X_0, \mathscr{O}_{X_0}(n))$ such that for all $n \geq n_0$, the closed subscheme $H_f = H_{f_0} \times_{S_0} S$ of X contains C as a closed subscheme and in addition $H_f \cap A = \emptyset$.[52]

[52] *Here f is the canonical image of f in $\Gamma(X, \mathscr{O}_X(n))$.*

Only the second condition of the conclusion remains to be verified. Take a generic point ξ of a positive-dimensional irreducible component of $F_{i,s}$, and write $s = p(s_0)$, $\xi_0 = p(\xi)$. Then there is an open neighborhood of ξ in $F_{i,s}$ not meeting C. Since $C = C_0 \times_{S_0} S$, the same is true for ξ_0 in $(F_{i,0})_s$. It follows that xi_0 is the generic point of a positive-dimensional irreducible component of $(F_{i,0} \backslash C_0)_s$, so that $\xi_0 \notin H_{f_0}$ and $\xi \notin H_f$. □

REMARK 4.9: The previous theorems are generalizations of a well-known classical result, namely the *avoidance lemma*: for a quasi-projective scheme X over a field, a closed subscheme C of X of positive codimension, and finitely many points $\xi_1, \ldots, \xi_n \in X$ not lying in C, there exists a hypersurface H in X containing C and avoiding $\xi_1, \ldots, \xi_n$.

4.2 *Existence of finite quasi-sections*

Given a morphism of schemes, a section might not exist in general, but in some arguments a weaker notion may suffice, namely that of a finite quasi-section.

DEFINITION 4.10: Let $X \to S$ be a surjective morphism of schemes. A *finite quasi-section* is a closed subscheme C of X such that $C \to S$ is finite and surjective. [53]

[53] *Cf. EGA IV, §14, p.200.*

EXAMPLE 4.11: Rational points of a variety over an algebraically closed field.

The results presented in the previous subsection can be used to establish the existence of finite quasi-sections for certain kind of morphisms.

THEOREM 4.12: Let S be an affine scheme and let $\pi : X \to S$ be a finitely presented projective morphism. Suppose that all the fibers of π are of the same dimension $d \geq 0$. Let C be a finitely presented closed subscheme of X such that $C \to S$ is finite. Then there exists a finitely presented finite quasi-section $T \to S$ containing C. Furthermore:

(a) Suppose that S is Noetherian and C, X are both irreducible. Then the quasi-section $T \to S$ can be chosen so that T is irreducible.

(b) Suppose that π is flat with fibers of constant pure dimension. Then the quasi-section $T \to S$ can be chosen to be flat.

Proof. Let us prove the first statement. It suffices to show the existence of a finitely presented finite quasi-section T for $X \to S$, for in that case $T \cup C$

is a finite quasi-section containing C which is obviously finitely presented. Afterwards, Theorem 4.8 is applied with $A, F = \emptyset$ and the existence of a hypersurface X is deduced. This hypersurfaces has already all the desired properties but the right dimension on the fibers, which however is one less than that of $X \to S$. So repeating the argument $d - 1$ times yields the desired quasi-section.

Now we turn to the additional conclusions.

(1) From the hypothesis it is clear that $X \to S$ is surjective and S is irreducible. Start with the case $d = 0$; then $X \to S$ itself is an irreducible finite quasi-section containing C as a closed subscheme. Now let $d \geq 1$. Then by 4.8 there exists a hypersurface H containing C as a closed subscheme. Using an additional technical lemma,[54] one finds an irreducible component Γ of H containing C, dominating S and such that $\Gamma \to S$ has all its fibers of dimension $d - 1$. Denote by $\mathscr{I}_C$ and $\mathscr{I}_\Gamma$ the sheaves of ideals defining C and Γ, respectively. Then $\mathscr{I}_\Gamma^m \subseteq \mathscr{I}_C$ for some $m > 0$. Thus we can endow the closed set Γ of X with the structure of a closed subscheme via $\mathscr{O}_X / \mathscr{I}_\Gamma^m$. By construction, we have that Γ is irreducible by construction and contains C as a closed subscheme. Thus we can iterate the process with $\Gamma \to S$.

(2) Analogous to (1). □

[54] *Cf. [GLL15], Lemma 6.4.*

4.3 *The final result*

THEOREM 4.13: Let A be a ring, and M a projective A-module of constant rank $r > 1$. There exists an A-algebra B that is finite and faithfully flat A, and such that $M \otimes_A B$ is isomorphic to a direct sum of projective B-modules of rank 1.

REMARK 4.14: The original statement assumes that M is finitely presented. However, this is not necessary, since it can be shown that every projective module of constant finite rank is finitely generated, hence finitely presented.

Proof. Let $S = \operatorname{Spec} A$. Consider the locally free $\mathscr{O}_S$-module $\mathscr{M}$ of rank r associated to M. Set $X = \mathbb{P}(\mathscr{M})$. Then we can apply Theorem 4.12 (3) to the canonical morphism $X \to S$ to deduce the existence of a finite flat quasi-section $f : T \to S$. We can write $T = \operatorname{Spec} B$ for some algebra B, finite and faithfully flat over A. But a morphism $T \to X$ corresponds to a surjective morphism $\alpha : f^*\mathscr{M} \twoheadrightarrow \mathscr{L}$ for some invertible $\mathscr{O}_T$-module $\mathscr{L}$. Then we have a decomposition $f^*\mathscr{M} \cong \mathscr{L} \oplus \mathscr{M}'$ with $\mathscr{M}' = \operatorname{Ker} \alpha$ locally free of rank $r - 1$. Thus iterating the same reasoning we obtain the desired splitting of $M \otimes_A B$ in rank 1 projective B-modules. □

EXAMPLE 4.15: We can exhibit some simple examples of this splitting property. If the ring A has the property that all finitely generated projective modules are free, then obviously we may take $B = A$. We already saw a slightly more sophisticated example in Section 1.5, in which a classification of projective modules over Dedekind domains was given. In this case also it suffices to take $B = A$, as theorem 1.46 shows.

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